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Understanding Topological Relationships through Comparisons of Similar Knots

Carol Strohecker

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Abstract

The results presented here come from a study in which learning about topology was supported by working with knots. In particular, comparisons of similar knots became a useful way for people to see the elements of the configurations and come to understand relationships among the parts.

This research is within the domain of mathematical thinking and is specifically concerned with how people develop understandings of the relationships of proximity that constitute topology. These relationships are characterized in terms of neighborhood, continuity, and boundaries. In combination with other basic conceptual structures – of classification, number, order, and seriations – the topological relationships are thought to give rise to understandings of mathematics.

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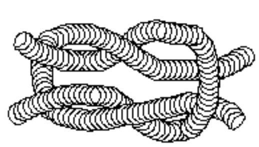
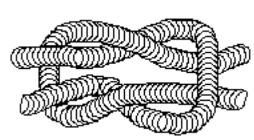
Studying people's thinking requires a setting conducive to in-depth exploration of the subjects' dialogue and actions. This need has implications for the duration of the study and for the nature of the environment in which the inquiry is conducted. The researcher must be able to spend a good deal of time with the subjects, encourage or design problem scenarios that will tend to bring out certain kinds of thinking, and develop a relationship with the subjects that allows for the exchange of relevant information.

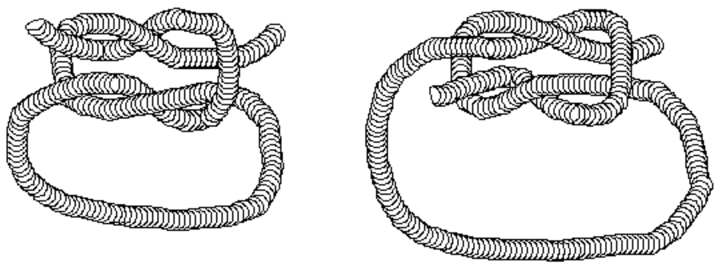
In this case, the subjects were 20 children aged 10 through 12. Over a period of 5 months, we developed an environment called the "Knot Lab" (Strohecker 1991, 1992, 1996). It consisted of knots and a social substrate that encouraged lively exchanges of ideas about them. The young people communicated with each other through various media, including constructions in string, pictures of knots, stories about knots, videos about knots, and large-scale displays of knots. Through these constructions, and through descriptions of the constructions, the young people expressed conceptions of the topological relationships embodied by various knots. We had some explicit discussions about these conceptions as the project completed. The discussions included direct comparisons of certain knots.

Comparing knots had become a useful way of seeing more clearly the elements of the configurations and coming to understand relationships among the parts. Likewise, consistencies from one knot to another formed a useful background against which differentiating features could become apparent. Especially useful were discussions of two knots that are strikingly similar, the Square knot and a variant of it called the Thief knot. These knots can be deceiving. In a localized view, the two knots seem identical:



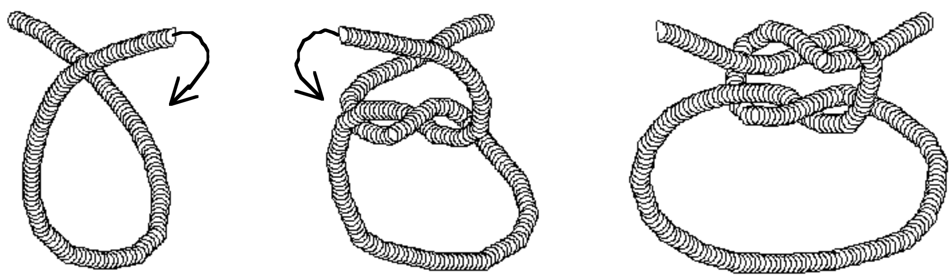


Yet, looking at the entire objects, it becomes clear that they are different:



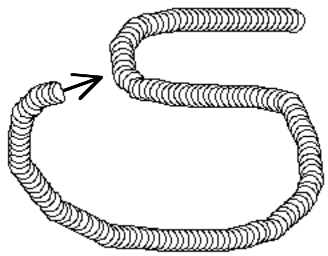
Square knot Thief knot

Appreciating the difference between these knots is related to understanding how they are produced. For this reason, comparisons of the Square knot and the Thief knot elicited understandings of knot-tying as well as of the completed configurations. Most of the children had learned to tie the Square knot using what we called the “upright” method:



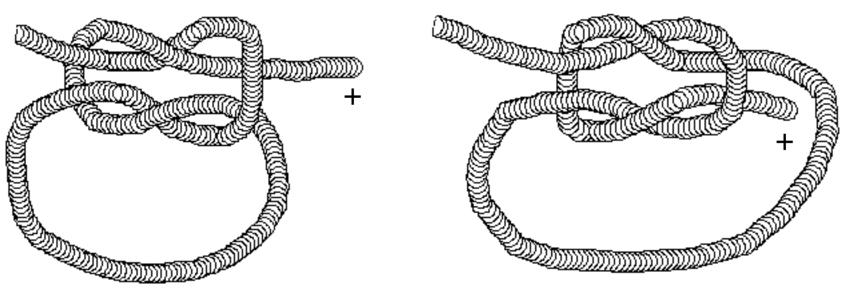
This approach to tying the Square and other knots had become customary, to the point of serving as a kind of default for many children as they began to learn a new knot. There is another fundamentally different approach to tying, though, which at first only a few of the children adopted. That approach involves thinking in terms of a single moving end that winds along a kind of pathway, wrapping around itself in certain places to produce the knot. The string becomes snake-like.

Interestingly, although both the Square knot and the Thief knot can be produced easily by this alternative method, some significant distortions of the upright method are required in order for it to produce a Thief knot. In fact, most children came to regard this effort as impossible: the Thief was tied in the snake-like manner or not at all. Many of them never managed to produce it, but those who did usually made some fundamental changes in their outlook and tying procedure. Such changes included laying the string flat on the table rather than holding it upright in two hands, and beginning the knot with some S-like pattern:



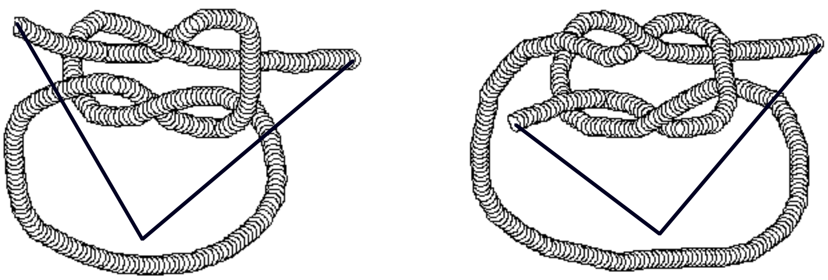
In the course of trying to figure out this approach, some children never quite reached the final solution, but struggled with the realization that the string had to be arranged in some new way. For others, this tying method became key to understanding that the Square and the Thief could not be the same knot. Of the 15 children who engaged in explicit comparisons of the two knots, 3 used tying as a technique in their reasoning and 4 others made some reference to the tying process without actively engaging it.

There are other ways of detecting differences, though. 13 children found a simple side-by-side comparison to be sufficient. They used terms like “up/down,” “top/bottom,” "above/below,” “over/under,” and “in/out” as they considered the positions of the ends relative to circular enclosure. These designations stemmed from thinking of the string as a boundary, dividing space into neighborhoods that are either in or outside of the knot.

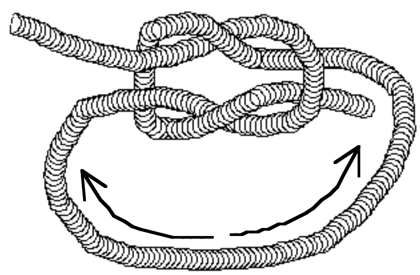


Eugene indicated his understanding of these relationships with gestures rather than words.[1] Making a “V” with the first and second fingers of one hand, he pointed to the ends in a peculiar way: the height of each finger was related to whether the end was above or below the enclosure.

[1] Pseudonyms have been substituted for the actual names of participants in the research.

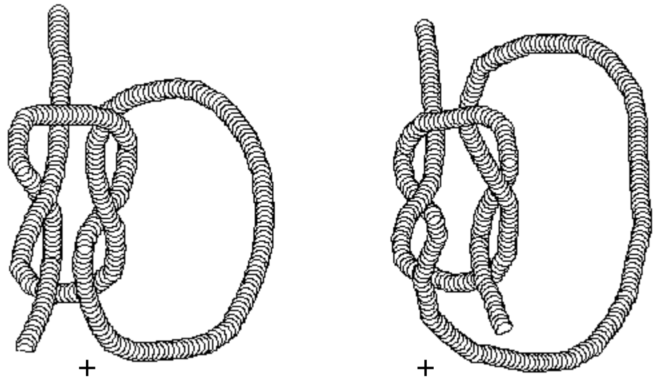


Stacy used a similar gesture as she tried to understand why she could not easily produce the Thief knot as she had the Square, through an upright method of tying. She indicated how the ends seemed to have proceeded upward, but found it difficult to understand how they then got "caught from in between."

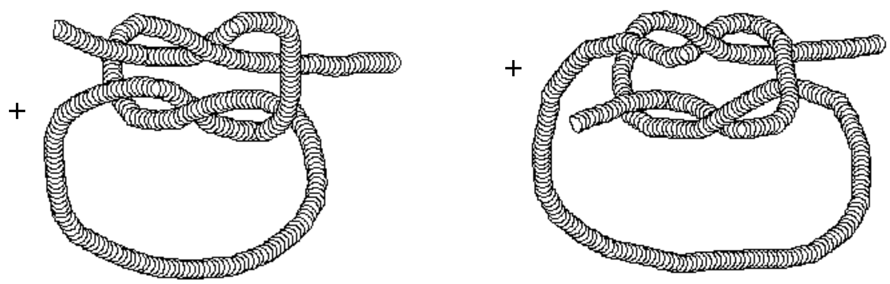


Many of the children involved their bodies in expressing their conceptions of knots and knot-tying, often relying on their arms or legs to represent ends of string moving into the form of a knot. The tactile and pliable qualities of the medium contributed to the "body syntonicity" that helped many children develop concrete understandings of the configurations (Papert 1980).

Most of the children oriented the knots with the entangled part at the top and the circular enclosure at the bottom, which is how the Square knot appears just after tying it with the upright method. However, a few children inverted the knots from this position, and sometimes they flipped the knots horizontally as well. Patrick flipped the Thief knot in this way, and then rotated both knots. These changes of view led to a breakthrough. He knew that something had looked "wrong" as he tried to match the two knots, but hadn't been able to see why until he changed the orientations:



Patrick’s revelation is interesting for its preservation of continuity between the entangled part and the circular enclosure of each knot. He focused on how the enclosure fed into the entanglement rather than viewing these parts as separate units, as most of the children tended to do. He was able to preserve the understanding in reorienting the knots:

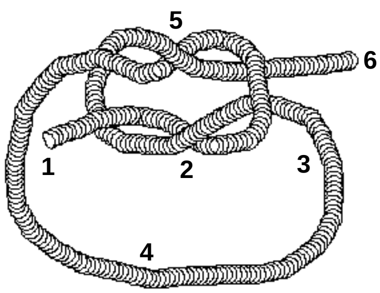


Making sense of how the string intertwines at the upper part of the knot can be difficult. These entanglements are just complicated enough to become frustratingly confusing at times. 7 of the children responded to my suggestion that they imagine themselves to be a tiny creature, like an ant, crawling along the knot. This Piagetian technique changes the scale of both space and time, enabling focus on difficult details so they can be sorted out (Piaget and Inhelder [1948, 1956] 1967).

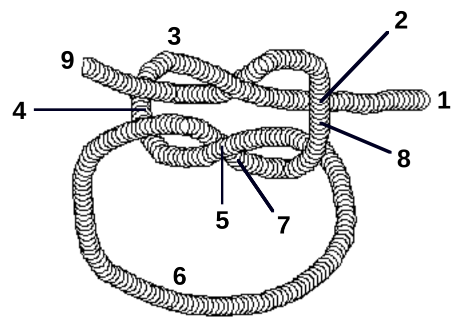
The technique serves to reveal senses of continuity and separation. It also reflects a way in which the knots can be tied, preserving the “over” and “under” moves in intertwining a piece of string. In the process of doing these tracings, 5 children went so far as to consider both faces of the knot, something none of those who had used only a side-by-side comparison had done. Some new concepts emerged, such as “bridges,” “blockades,” and “U-curves” of two interlocking loops.

Maria was only partially successful (below left). She made her way past the first two crossings but lost track at the third, going downward into the enclosure rather than upward into the entanglement. She broke continuity and jumped from one strand to the other, creating a shortcut that enabled her to simplify the path, avoiding the intertwined loops altogether. By ignoring complications, she found a quick, albeit inaccurate, route out of the knot.

Alice, though, was able to maintain continuity in following the ant's path (below right). She dwelt more on details and took care to describe her route. She invented the term "bridge" at the first crossing (2), invoked it again at (4) even though it was then in the lower plane, and used it again at (7), even though this bridge had no coupled crossing in the lower plane.

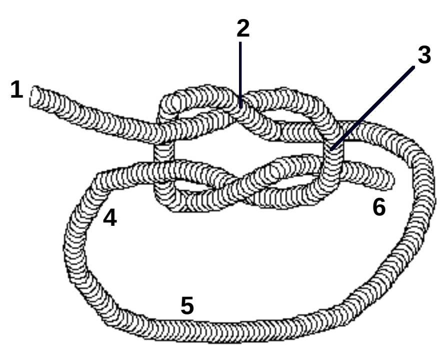


Maria's tracing of the Thief knot

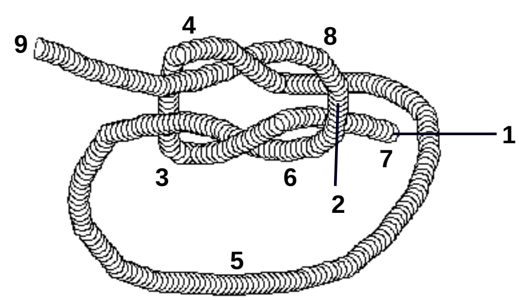


Alice's tracing of the Square knot

Because of this attention to the crossings, Alice was able to trace the Thief knot correctly as well. Patrick also managed it, in spite of being disturbed by the interruption of continuity at the first crossing. Whereas Alice dubbed that separation a "bridge" and blithely hopped over it, Patrick called it a "blockade" but managed not to let it get in his way. He stayed on track until the end, when he nearly exited the knot too early (7), but quickly caught himself and then completed the pathway correctly.



Alice's tracing of the Thief knot



Patrick's tracing of the Thief knot

Discussing the microstructure of an artifact such as a knot can help in recapturing the way in which the object was made and in grasping how its components interrelate. Through conversations such as these, children arrived at understandings of the relationships that characterize certain knots and differentiate one knot from another. The children dealt explicitly with the topological relationships of neighborhood, continuity, and boundaries. Their understandings were often quite robust, surviving variations in the form or setting of a knot.

The environment that enabled conversations at this level of detail emphasized producing knots, considering the artifact that had been created as well as how it had been made, and trying to explain these considerations to someone else. Situating such explanations in conversations helped to promote an awareness of one’s own processes of thinking and doing. Producing, communicating, and self-reflecting are important features of a research environment, but they are also powerful tools for learning.

Acknowledgments

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