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Net-Voltage Screening of Bivariate-Bicycle Tanner-Graph Covers

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***Abstract*—Finite-length bivariate-bicycle (BB) quantum LDPC codes are attractive low-overhead memory candidates, but typical design methods consider only a scalar cover factor. We introduce a directionally shifted-cover design rule that treats both the split $h = a_x a_y$ and the monomial lift offsets as optimization variables. The novelty is a short-cycle net-voltage criterion. Support-difference equality defines a base Tanner 4-cycle in the base polynomial pair, which is assigned voltages in $G_h = \mathbb{Z}_{a_x} \times \mathbb{Z}_{a_y}$; zero voltage lets the corresponding small base Tanner cycle or closed walk return to the starting lifted vertex after one traversal, whereas nonzero high-order voltage forces it to close after r traversals, where r is the net-voltage order. This criterion provides an interpretable prefilter before exact distance verification and explains why cover direction matters at fixed length. For a $(6, 6)$ base and $h = 4$, the optimized pure y-cover gives $[[288, 20, 18]]$, improving kd^2/n over the optimized pure x-cover $[[288, 16, 18]]$.**

I. Introduction

Quantum low-density parity-check (qLDPC) codes seek sparse stabilizer checks together with high rate and large distance, making them attractive candidates for scalable fault-tolerant memory. Product and graph-based constructions have driven much of the progress: hypergraph-product codes gave positive-rate qLDPC families with distance growing as the square root of block length [1], while balanced-product, lifted-product, and quantum Tanner constructions substantially improved the asymptotic landscape [2]–[5]. For near- and medium-term applications, however, finite-length performance remains just as important as asymptotic scaling, and degenerate qLDPC codes have shown strong performance under belief-propagation with ordered-statistics decoding (BP-OSD) [6].

Bivariate-bicycle (BB) codes have recently become a leading finite-length code family because they combine compact polynomial descriptions, sparse checks, and favorable layouts. Bravyi *et al.* used BB codes for high-threshold, low-overhead quantum memory [7]. Subsequent work expanded the same polynomial framework through multivariate bicycle codes [8], lower-connectivity BB measurement circuits [9], symmetry and logical-gate structure [10], and BB code families from covering graphs [11]. In particular, h-cover BB codes show that many useful codes can be understood as lifts of smaller Tanner graphs.

This paper refines the covering-graph viewpoint for finite-length BB-code selection. For a fixed cover degree h, we choose positive integers a_x, a_y satisfying $h = a_x a_y$, which scale the base periods as $(\ell, m) \longmapsto (a_x \ell, a_y m)$. We refer to (a_x, a_y) as the cover factorization. Different factorizations have the same block-length scaling but can produce different dimensions and distances. The monomial lift offsets determine the lifted polynomial exponents and thereby induce a voltage assignment on the base Tanner graph. Our contribution is a short-cycle net-voltage screening criterion: assignments are ranked according to the net voltages of selected short base cycles, after which the dimension and minimum distance are evaluated exactly.

II. Background and Cover Construction

A BB code is a two-block CSS code generated by two sparse polynomials in the quotient ring

$$R = \mathbb{F}_2[x, y]/(x^\ell - 1,\, y^m - 1),$$

where exponent pairs are read on the torus $\mathbb{Z}_\ell \times \mathbb{Z}_m$. Let

$$P = \sum_{i=1}^{d_P} x^{a_i} y^{b_i}, \qquad Q = \sum_{j=1}^{d_Q} x^{c_j} y^{e_j}.$$

If $A = P(x, y)$ and $B = Q(x, y)$, the usual BB checks are

$$H_X = [A\ B], \qquad H_Z = [B^T\ A^T].$$

The commutation condition follows from $AB + BA = 0$ over $\mathbb{F}_2$, so the base length is $n_0 = 2\ell m$.

For an h-fold cover, choose positive integers a_x, a_y satisfying $h = a_x a_y$. The lifted ring is

$$R_h = \mathbb{F}_2[x, y]/(x^{a_x \ell} - 1,\, y^{a_y m} - 1),$$

which sends (ℓ, m) to $(a_x \ell, a_y m)$ and gives $n_h = h n_0$. A monomial receives a monomial lift offset $(s, t) \in \mathbb{Z}_{a_x} \times \mathbb{Z}_{a_y}$ and is displaced by

$$x^a y^b \longmapsto x^{a+\ell s} y^{b+mt},$$
$$s \in \{0, \ldots, a_x - 1\}, \quad t \in \{0, \ldots, a_y - 1\}.$$

Applying this map to every term gives $\widetilde{P}, \widetilde{Q} \in R_h$, and the lifted CSS code is formed from the same matrices as above.

III. Net Voltages of Base Tanner-Graph Cycles

Here a_i, b_i are the exponent coordinates of the i-th monomial $x^{a_i} y^{b_i}$ in P, and c_j, e_j are the corresponding exponent coordinates of the j-th monomial $x^{c_j} y^{e_j}$ in Q. After the construction is fixed, each monomial shift is viewed as a voltage in

$$G_h = \mathbb{Z}_{a_x} \times \mathbb{Z}_{a_y}, \qquad \gamma_{P,i} = (s_i, t_i), \quad \gamma_{Q,j} = (s'_j, t'_j).$$

For two monomials of P and two monomials of Q, define

$$\Delta_P(i, i') = (a_i - a_{i'},\, b_i - b_{i'}) \pmod{(\ell, m)},$$
$$\Delta_Q(j, j') = (c_j - c_{j'},\, e_j - e_{j'}) \pmod{(\ell, m)}.$$

TABLE I
CODES FROM THE EXAMPLE 1 BASE.

Code	(ℓ, m)	n	k	d	$\widehat{d}_D(\sigma)$	Cover	Shift
D_0	$(6,6)$	72	16	6	4	base	no shift
D_1	$(12,6)$	144	16	12	4	2	$s_P=(0,0,1,0),\ s_Q=(0,0,0,0)$
D_1'	$(6,12)$	144	20	10	4	2	$t_P=(0,1,0,1),\ t_Q=(0,0,1,0)$
D_2	$(24,6)$	288	16	18	4	4	$s_P=(0,2,3,1),\ s_Q=(0,3,3,2)$
D_2'	$(6,24)$	288	20	18	8	4	$t_P=(0,1,0,1),\ t_Q=(0,0,1,2)$

A mixed-block base 4-cycle is the equality $\Delta_P(i,i') = \Delta_Q(j,j')$. Its lifted voltage is

$$\Omega(i,i';j,j') = (\gamma_{P,i} - \gamma_{P,i'}) - (\gamma_{Q,j} - \gamma_{Q,j'}) \in G_h.$$

The design objective is not merely to make these voltages nonzero, but to make their group orders as large as possible, so that small base Tanner-graph cycles (cycles) must wind through many fibers before closing. This formula is the link between the construction and the search rule. If $\Omega = 0$, the repeated displacement closes inside one fiber and can support a small lifted dependency or logical candidate. If $\Omega \neq 0$, one traversal changes the fiber coordinate or group coordinate by Ω, so the structure closes only after the first return to zero. For $\Omega = (\Omega_x, \Omega_y)$, that return order is

$$\mathrm{ord}_{G_h}(\Omega) = \mathrm{lcm}\left(\frac{a_x}{\gcd(a_x,\Omega_x)}, \frac{a_y}{\gcd(a_y,\Omega_y)}\right),$$

with $\gcd(a,0) = a$. For a target distance D, let $\mathcal{C}_{<D}$ be the small base Tanner-graph cycles that may lift to objects below weight D. For a shift assignment σ, use the prefilter

$$\widehat{d}_D(\sigma) = \min_{C \in \mathcal{C}_{<D}} w(C)\, \mathrm{ord}_{G_h}(\Omega_\sigma(C)),$$

$$\widehat{S}(\sigma) = \frac{k(\sigma)\widehat{d}_D(\sigma)^2}{hn_0}.$$

We first assign a nonidentity net voltage to the selected base cycle by assigning nonzero high-order voltage, then choose the directional split that preserves larger k, and finally verify the exact distance d. Here $\widehat{d}_D$ and $\widehat{S}$ are only screening quantities used to rank shifts before exact distance verification.

IV. NUMERICAL RESULTS

Example 1. *Let* $R = \mathbb{F}_2[x,y]/(x^6-1,\ y^6-1)$, $P = y^5 + x^2y^5 + x^3y + x^4y$, *and* $Q = x + xy^5 + x^2y^2 + x^3y^3$. *The search covers are listed in Table I.*

For D_2', the cover is $(6,6) \mapsto (6,24)$, so $(a_x,a_y) = (1,4)$ and $G_h \cong \mathbb{Z}_4$. Label the monomials by $P_1 = y^5$, $P_2 = x^2y^5$, $P_3 = x^3y$, $P_4 = x^4y$, and $Q_1 = x$, $Q_2 = xy^5$, $Q_3 = x^2y^2$, $Q_4 = x^3y^3$. The mixed-block base 4-cycles are

$$\Delta_P(1,4) = \Delta_Q(4,2), \qquad \Delta_P(2,3) = \Delta_Q(1,3),$$
$$\Delta_P(3,2) = \Delta_Q(3,1), \qquad \Delta_P(4,1) = \Delta_Q(2,4).$$

Under the D_2' shifts, their voltages are $1,2,2,3$ in $\mathbb{Z}_4$. None are zero, 1 and 3 have order four, and 2 has order two. Hence these selected short base cycles do not close in one fiber, which explains why the pure y-cover is structurally strong and why it attains the best score in the table.

Figure 1 compares the $D_2 = [[288,16,18]]$ code, which has voltage prefilter $\widehat{d}_D = 4$, with the $D_2' = [[288,20,18]]$ code, which has $\widehat{d}_D = 8$. A larger prefilter value can indicate a structurally better shifted cover before exact distance verification.

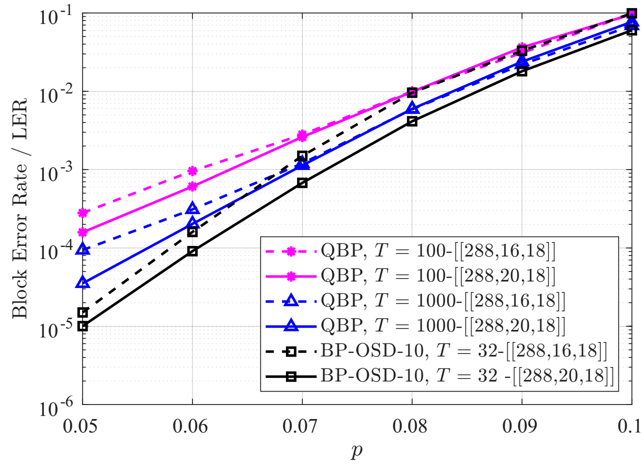

Fig. 1. Block error rate for the constructed $[[288,20,18]]$ and $[[288,16,18]]$ codes. QBP and BP-OSD decoders were allowed T iterations.

V. CONCLUSION

The cover split $h = a_xa_y$ first defines the lifted BB geometry, and the voltage shifts then determine how small base Tanner-graph cycles behave in that geometry. Nonzero high-order net voltage of the corresponding base cycle gives an interpretable screen for promising finite-length covers before exact distance computation. The examples show that the cover direction matters at fixed h: for $h = 4$, D_2' reaches the same verified distance as the pure x-cover D_2 while preserving four additional logical qubits. Future work will combine this voltage filter with automated exact-distance verification for larger BB searches.

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REFERENCES

[1] J.-P. Tillich and G. Zémor, "Quantum LDPC codes with positive rate and minimum distance proportional to the square root of the blocklength," *IEEE Trans. Inf. Theory*, vol. 60, no. 2, pp. 1193–1202, 2014.
[2] N. P. Breuckmann and J. N. Eberhardt, "Balanced product quantum codes," *IEEE Trans. Inf. Theory*, vol. 67, no. 10, pp. 6653–6674, 2021.
[3] P. Panteleev and G. Kalachev, "Quantum LDPC codes with almost linear minimum distance," *IEEE Trans. Inf. Theory*, vol. 68, no. 1, pp. 213–229, 2022.
[4] P. Panteleev and G. Kalachev, "Asymptotically good quantum and locally testable classical LDPC codes," in *Proc. ACM STOC*, 2022, pp. 375–388.
[5] A. Leverrier and G. Zémor, "Quantum Tanner codes," in *Proc. IEEE FOCS*, 2022, pp. 872–883.
[6] P. Panteleev and G. Kalachev, "Degenerate quantum LDPC codes with good finite length performance," *Quantum*, vol. 5, p. 585, 2021.
[7] S. Bravyi, A. W. Cross, J. M. Gambetta, D. Maslov, P. Rall, and T. J. Yoder, "High-threshold and low-overhead fault-tolerant quantum memory," *Nature*, vol. 627, pp. 778–782, 2024.
[8] L. Voss, S. J. Xian, T. Haug, and K. Bharti, "Multivariate bicycle codes," *Phys. Rev. A*, vol. 111, p. L060401, 2025.
[9] M. H. Shaw and B. M. Terhal, "Lowering connectivity requirements for bivariate bicycle codes using morphing circuits," *Phys. Rev. Lett.*, vol. 134, p. 090602, 2025.
[10] J. N. Eberhardt and V. Steffan, "Logical operators and fold-transversal gates of bivariate bicycle codes," *IEEE Trans. Inf. Theory*, vol. 71, no. 2, pp. 1140–1152, 2025.
[11] B. C. B. Symons, A. Rajput, and D. E. Browne, "Sequences of bivariate bicycle codes from covering graphs," arXiv:2511.13560, 2025.