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Base-Preserving APM/Voltage Lifts of Bivariate Bicycle Quantum LDPC Codes

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***Abstract*—Bivariate-bicycle (BB) quantum LDPC codes are compact Calderbank–Shor–Steane (CSS) codes with sparse checks and strong finite-length performance. Standard BB codes represent each monomial $x^a y^b$ by a two-coordinate circulant permutation matrix, so that the CSS condition follows from commutation of the x and y shifts. We summarize a base-preserving affine-permutation/voltage-lift extension of BB codes. Each base monomial is replaced by a lifted monomial acting on a three-coordinate permutation. The novelty is the BB-compatible use of these lifted monomials. We demonstrate that the voltage lifting can effectively remove short cycles.**

I. Introduction

Quantum LDPC codes seek sparse stabilizer checks together with high rate and large distance. Product-type constructions such as hypergraph-product, lifted-product, and quantum Tanner codes have driven major asymptotic progress [1]–[4]. For finite-length practical systems, bivariate-bicycle (BB) codes have become a leading family because they admit compact polynomial descriptions, sparse degree-six checks, and hardware-oriented layouts: e.g., for a high-threshold, low-overhead memory proposal [5]. Subsequent work has studied BB circuits, logical structures, and cover extensions [6]–[8].

Most BB constructions use commuting circulant permutation matrix (CPM) shifts. While CPM and voltage-lifts are widely used [9], affine permutation matrix (APM) can further extend them [10]. Recent quantum work has also used controlled commutativity of permutation matrices to increase Tanner girth under CSS orthogonality constraints [11]. The contribution of this work is a BB-compatible, base-preserving APM/voltage formulation in which each monomial $x^a y^b$ is augmented by affine voltage parameters on an additional cyclic sheet coordinate. This retains the two-block CSS structure $H_X = [A\ B], H_Z = [B^T\ A^T]$, while enforcing the lifted commutation condition $AB + BA = 0$. The resulting design freedom, where the intended algebraic support and qubit connectivity follows the BB base and the voltage assignment changes how the base edges move between sheets, can break short cycles without changing the BB polynomial support, allowing cycle structure, distance, rate, and decoder performance to be optimized within the BB framework.

II. BB Codes and Affine Monomials

A weight-six BB code is specified by appropriate weight polynomials $P = \sum_i x^{a_i} y^{b_i}$ and $Q = \sum_j x^{c_j} y^{d_j}$ over $R = \mathbb{F}_2[x, y]/(x^\ell - 1, y^m - 1)$. Setting $A = P(x, y)$ and $B = Q(x, y)$, we construct

$$H_X = [A\ B],\ H_Z = [B^T\ A^T],\ H_X H_Z^T = AB + BA. \quad (1)$$

For ordinary BB codes, $AB+BA = 0$. Here, we use the CPM exponents (a, b) of monomial $x^a y^b$ to write

$$\mathcal{M}(a, b; \lambda, \mu; e) : (i, j, s) \mapsto (i+a,\ j+b,\ s+\lambda i+\mu j+e), \quad (2)$$

on $\Omega = \mathbb{Z}_\ell \times \mathbb{Z}_m \times \mathbb{Z}_q$, where λ is the fiber from the x-coordinate i, μ is the fiber from the y-coordinate j, e is a constant fiber shift offset from sheet coordinate s, and the coordinates are reduced modulo ℓ, m, q, respectively. $\mathcal{M}(a, b; \lambda, \mu; e)$ defines an affine permutation matrix, in the spirit of APM-LDPC [10]. The lifting is then defined using

$$A = \textstyle\sum_i \mathcal{M}(a_i, b_i; \lambda_i, \mu_i; e_i),\ B = \sum_j \mathcal{M}(c_j, d_j; \alpha_j, \beta_j; f_j),$$

for similarly defined α, β, f. We note that this is a more general lifting than the conventional CPM construction. For example, the conventional CPM lifting can be described by monomials of the form $\mathcal{M}(a, b; 0, 0; 0)$.

III. CSS and Girth 8 Conditions

For two monomials $\mathcal{M}(a, b; \lambda, \mu; e)$ and $\mathcal{M}(c, d; \alpha, \beta; f)$, a sufficient pairwise commutation check is

$$\lambda c + \mu d \equiv \alpha a + \beta b \pmod q. \quad (3)$$

The construction follows the BB product condition $AB + BA = 0$, which is equivalent to $H_X H_Z^T = 0$ for the two-block CSS matrices. The girth 8 condition is imposed through optimization via the APM conditions. let $T_t = \mathcal{M}(a, b; \lambda, \mu; e)$ be the permutation in (2) that corresponds to an edge t in the base graph. Its inverse is

$$T_t^{-1}(i, j, s) = (i - a,\ j - b,\ s - \lambda(i - a) - \mu(j - b) - e).$$

Let $\Gamma_{2r}(H)$ be the set of length-$2r$ closed walks in the block Tanner graph (base graph) of H. It is easy to see that cycles in the lifted graph can only arise as lifts of cycles in the base graph. For $C = (t_1, \ldots, t_{2r}) \in \Gamma_{2r}(H)$ define $W_C = T_{t_{2r}}^{-1} T_{t_{2r-1}} \cdots T_{t_2}^{-1} T_{t_1}$, where the rightmost map acts first. The walk becomes a cycle in the lifted Tanner graph iff W_C has a fixed point. Since the maps are affine-voltage maps,

$$W_C(i, j, s) = (i + \Delta_x,\ j + \Delta_y,\ s + \Lambda i + \Theta j + \omega),$$

TABLE I
SELECTED CPM AND APM/VOLTAGE BB EXAMPLES. IN THE A AND B COLUMNS, $(a, b; \lambda, \mu; e)$ DENOTES $\mathcal{M}(a, b; \lambda, \mu; e)$.

A	B	(ℓ, m, q)	Code	(N_4, N_6, N_8)
$(3, 0; 0, 0; 0), (0, 1; 1, 0; 0), (0, 2; 0, 0; 0)$	$(0, 3; 0, 0; 0), (1, 0; 0, 1; 0), (2, 0; 0, 0; 0)$	$(6, 6, 2)$	$[[144, 12, 6]]$	$(0, 0, 2232)$
$(3, 0; 0, 0; 0), (0, 1; 1, 0; 0), (0, 2; 0, 0; 0)$	$(0, 3; 0, 0; 0), (1, 0; 0, 1; 0), (2, 0; 0, 0; 0)$	$(12, 6, 2)$	$[[288, 12, 12]]$	$(0, 0, 2304)$
$(3, 0; 0, 0; 0), (0, 1; 0, 0; 0), (0, 2; 0, 0; 0)$	$(0, 3; 0, 0; 0), (1, 0; 0, 0; 0), (2, 0; 0, 0; 0)$	$(12, 6, 1)$	$[[144, 12, 12]]$	$(0, 144, 1728)$
$(1, 0; 0, 1; 1), (2, 0; 0, 0; 1), (3, 3; 0, 0; 0)$	$(2, 5; 1, 1; 0), (1, 1; 1, 1; 1), (3, 2; 0, 0; 1)$	$(6, 8, 2)$	$[[192, 12, 8]]$	$(0, 0, 1872)$
$(5, 4; 1, 1; 1), (1, 2; 1, 1; 0), (3, 4; 1, 1; 0)$	$(2, 3; 1, 0; 1), (0, 0; 0, 0; 0), (1, 4; 1, 0; 0)$	$(6, 6, 2)$	$[[144, 8, 12]]$	$(0, 96, 1656)$

where Δ_x, Δ_y are accumulated base shifts, Λ, Θ are accumulated coefficients of the starting coordinates i, j, and ω is the accumulated constant voltage. Thus we have 8-girth when $\mathrm{Fix}(W_C) = \varnothing$ and $C \in \Gamma_4(H_X) \cup \Gamma_4(H_Z) \cup \Gamma_6(H_X) \cup \Gamma_6(H_Z)$. Equivalently, each such C must either have 1) $\Delta_x \not\equiv 0 \pmod{\ell}$ or 2) $\Delta_y \not\equiv 0 \pmod{m}$, or, 3) if the base coordinates pass, the sheet equation $\Lambda i + \Theta j + \omega \equiv 0 \pmod{q}$ has no solution $(i, j) \in \mathbb{Z}_\ell \times \mathbb{Z}_m$. In the pure-voltage case, this reduces to $\omega \not\equiv 0 \pmod{q}$.

a) Example: Consider $[[72, 12, 6]]$ BB code with $P = x^3 + y + y^2$ and $Q = y^3 + x + x^2$. For an alternative $q = 2$ lift with $(\ell, m) = (6, 6)$ (that gives double the length), choose

$$A = \mathcal{M}(3, 0; 0, 0; 0) + \mathcal{M}(0, 1; 1, 0; 0) + \mathcal{M}(0, 2; 0, 0; 0),$$
$$B = \mathcal{M}(0, 3; 0, 0; 0) + \mathcal{M}(1, 0; 0, 1; 0) + \mathcal{M}(2, 0; 0, 0; 0).$$

Note that the middle permutations are the only ones that change: y is instead lifted as $\mathcal{M}(0, 1; 1, 0; 0)$ and x is instead lifted as $\mathcal{M}(1, 0; 0, 1; 0)$. These two voltage choices preserve commutation by (3) because $1 \cdot 1 + 0 \cdot 0 = 0 \cdot 0 + 1 \cdot 1$ in $\mathbb{Z}_2$. They also change the sheet coordinate around short base walks, removing all 4- and 6-cycles. The checks are $H_X = [A\ B]$ and $H_Z = [B^T\ A^T]$ with $n = 2\ell mq = 144$. We constructed the first three rows of Table I from this base example.

IV. RESULTS AND DISCUSSION

Table I lists some lifted A and B examples with corresponding code parameters and number of cycles of length w. The first two rows use the same voltage pattern at different base sizes, both achieving girth 8 but requiring either longer length or achieving less distance than the benchmark. The third row is the CPM BB benchmark, with all voltage entries zero, but with girth 6. The fourth row shows an example with the best cycle distribution but distance 8 and requiring a slightly longer code length. The last row gives a lower-rate $[[144, 8, 12]]$ example with fewer 6-cycles than the benchmark.

Fig. 1 presents numerical simulations for three length-144 codes with quaternary belief propagation (QBP) and BP-OSD-10 decoders. The girth 8, $[[144, 12, 6]]$ code has the weakest curve because its distance is only 6, showing that cycle removal alone cannot replace distance. The $[[144, 8, 12]]$ example has the lowest BP-OSD-10 error rate over the plotted range and remains competitive under QBP, indicating that lowering k and improving the lifted graph can produce useful finite-length behavior. The CPM $[[144, 12, 12]]$ benchmark remains strong because it preserves both $k = 12$ and distance 12, although it has the worst cycle spectrum.

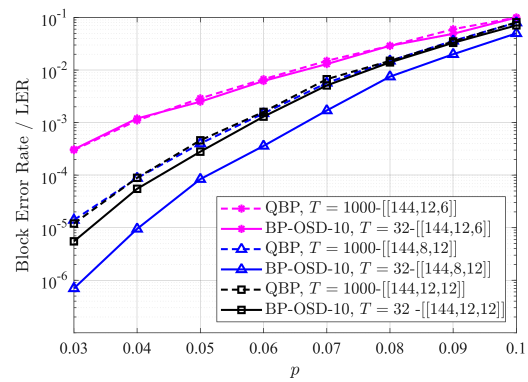

Fig. 1. Block error rate / logical error rate (LER) versus physical error probability p for the girth 8 $[[144, 12, 6]]$ code, the lower-rate $[[144, 8, 12]]$ voltage example, and the CPM $[[144, 12, 12]]$ benchmark.

V. CONCLUSION

A base-preserving APM/voltage lift gives a BB-compatible way to add structured design freedom while retaining the two-block CSS form. The examples show that the voltage layer can remove short Tanner cycles and improve finite-length behavior when optimized together with distance, rate, and decoding performance. The $[[144, 8, 12]]$ example demonstrates competitive BP-OSD performance at the same block length. These results suggest that APM/voltage lifting is a promising direction for finite-length BB quantum LDPC code design. Future work will automate the search over voltage parameters.

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