MITSUBISHI ELECTRIC RESEARCH LABORATORIES
https://www.merl.com

Relaxed barrier function based model predictive control with hard input constraints

Castroviejo-Fernandez, Miguel; Leung, Jordan

TR2026-005 January 07, 2026

Abstract

This letter focuses on a formulation of Model Predictive Control (MPC) with an optimal control problem (OCP) defined by hard input constraints and soft state and terminal set constraints. The soft constraints are accounted for as relaxed barrier function terms in the objective function. The proposed MPC is feasible for any state vector and, assuming the input constraint set is simple (e.g. a hyperrectangle), leads to anytime feasible formulations. A theoretical description of the MPC scheme is conducted. Among other results, asymptotic stability of the proposed MPC is proven and a region of attraction (RoA) estimate is derived. Moreover, stability guarantees when performing a limited number of optimization iterations are also derived. Numerical results showcase the benefit of considering the input constraints directly in the OCP instead of saturating the output of an unconstrained OCP with relaxed barrier functions, as was previously done in the literature.

IEEE Control Systems Letters 2025

Mitsubishi Electric Research Laboratories, Inc.
201 Broadway, Cambridge, Massachusetts 02139

Relaxed barrier function based model predictive control with hard input constraints

Miguel Castroviejo-Fernandez[1], Jordan Leung[2, ⋆]

***Abstract*— This letter focuses on a formulation of Model Predictive Control (MPC) with an optimal control problem (OCP) defined by hard input constraints and soft state and terminal set constraints. The soft constraints are accounted for as relaxed barrier function terms in the objective function. The proposed MPC is feasible for any state vector and, assuming the input constraint set is simple (e.g. a hyperrectangle), leads to anytime feasible formulations. A theoretical description of the MPC scheme is conducted. Among other results, asymptotic stability of the proposed MPC is proven and a region of attraction (RoA) estimate is derived. Moreover, stability guarantees when performing a limited number of optimization iterations are also derived. Numerical results showcase the benefit of considering the input constraints directly in the OCP instead of saturating the output of an unconstrained OCP with relaxed barrier functions, as was previously done in the literature.**

I. Introduction

Model predictive control (MPC) has established itself as a popular formalism for the control of constrained systems due to its direct constraint handling capabilities and performance guarantees. MPC is implemented by solving, at each time instant, an optimal control problem (OCP) that minimizes an objective function over a finite input sequence subject to the system dynamics and constraints. The OCP is parameterized by the current state vector and the input is selected as the first element of the optimal input sequence. Stability and recursive feasibility guarantees for state and input constrained MPC are readily available, e.g. [1].

Nevertheless, unforeseen disturbances, sensor noise, or the use of a suboptimal solution can lead to an unexpected departure of the state from the feasible region of the OCP leading to an undefined control input [2]. Considerable effort has been given to designing MPC schemes which are feasible over the whole state space. A popular approach is to use slack variables to relax the constraints and modify the objective [3]–[6]. Slack variables form a natural relaxation strategy, but the relaxed OCP typically still has constraints that preclude the use of unconstrained or projection-based optimization, which complicates the implementation of suboptimal MPC.

A different relaxation of MPC [7] is based on using relaxed barrier functions to replace both state and input constraints in the OCP. This leads to an unconstrained, (at least) twice continuously differentiable, and strongly convex program that can be solved using sparsity-exploiting algorithms [8]. In [7], the authors derive global asymptotic stability results at the expense of arbitrarily large input constraint violations. While the approach is appealing, in practice, input constraints often represent physical limitation on the actuators. As such, the input produced by [7] must be saturated after its computation, which may cause performance degradation or loss of stability.

In this work, we modify the relaxed barrier MPC formulation of [7] to directly account for the input constraints in the OCP. More precisely, we propose an MPC scheme with hard input constraints and soft state and terminal state constraints encoded as relaxed barrier function terms in the OCP. The resulting OCP is (at least) twice continuously differentiable, strongly convex, and feasible for any value of the state vector. In addition, if the input constraint set is simple, e.g. a hyperrectangle, it can be readily solved using projected Newton methods [9] and a feasible solution can be obtained after any number of iterations. We derive theoretical stability guarantees for this MPC formulation under both optimal and suboptimal solutions of the OCP. A numerical study showcases the advantage of the proposed MPC compared to that of [7] when strict enforcement of input constraints is necessary.

In the remainder of this section we introduce some notation and the concept of relaxed barrier functions. Section II describes the problem at hand and defines the OCP we consider. In Section III theoretical guarantees are derived and Section IV presents numerical results.

A. Notation

Let $\mathbb{Z}$ be the set of integers and $\mathbb{R}$ the set of real numbers. Given sets $\mathcal{S}, \mathcal{A} \subseteq \mathbb{R}$, then $\mathcal{S}_{\mathcal{A}} \triangleq \mathcal{S} \cap \mathcal{A}$ and for $a \in \mathbb{R}$, $\mathcal{S}_{\geq a} \triangleq \mathcal{S}_{[a,\infty)}$ and $\mathcal{S}_{>a} \triangleq \mathcal{S}_{(a,\infty)}$. The interior and boundary of a set $\mathcal{S} \subset \mathbb{R}^n$ are denoted as $\mathcal{S}^\circ$ and $\partial\mathcal{S}$, respectively. Moreover, $a\mathcal{S} \triangleq \{as : s \in \mathcal{S}\}$, for all $a \in \mathbb{R}$. Finite sequences are written in bold font with a subscript denoting their length, i.e., $\boldsymbol{a}_N$ denotes the sequence $\{a_i\}_{i=0}^{N-1}$. With a slight overloading of notation, we also use $\boldsymbol{a}_N$ to denote the set containing elements a_i for $i = 0, .., N-1$. For a given matrix $A \in \mathbb{R}^{n\times m}$ its i^{th} row is denoted by $A_{(i)}$.

B. Relaxed barrier functions

Throughout this work we will make use of relaxed barrier functions, which are barrier functions extended to be defined on the whole underlying vector space [10]. We introduce two specific instances of relaxed barrier functions that are based on a quadratic relaxing function:

$$\mathbb{R} \to \mathbb{R} : s \to \beta(s;\delta) \triangleq \frac{1}{2}\left[\left(\frac{s-2\delta}{\delta}\right)^2 - 1\right] - \ln(\delta), \quad (1)$$

[1]Department of Aerospace Engineering, the University of Michigan, mcastrov@umich.edu. This work was completed while M. Castroviejo-Fernandez was an intern at MERL.

[2, ⋆] Corresponding author, Mitsubishi Electric Research Laboratories (MERL), leung@merl.com

where $\delta \in \mathbb{R}_{>0}$ is the relaxation parameter. We note that β is strictly monotone and a continuously differentiable function. Importantly for the relaxed logarithmic barrier functions $\beta(\delta;\delta) = -\ln(\delta)$ and $\beta \to \infty$ as $s \to -\infty$.

Definition 1. *Let $\beta(\cdot;\delta)$ be defined as in (1). Then, $\mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{\infty\} : s \to \hat{B}(s;\delta)$,*

$$\hat{B}(s;\delta) \triangleq \begin{cases} -\ln(s), & s > \delta \\ \beta(s;\delta), & s \leq \delta, \delta > 0 \\ \infty, & s \leq \delta, \delta = 0 \end{cases}, \tag{2}$$

is a quadratically relaxed logarithmic barrier function.

The relaxed barrier function in Definition 1 is useful to relax constraints represented as sublevel sets of a functional. Also, the function $\hat{B}(\cdot;0)$ recovers the natural logarithm, i.e., the case of zero relaxation. A similar observation applies to Definition 2, which introduces a relaxation for polytopic set constraints.

Definition 2. *Let β be defined in (1), $\hat{B}$ in (2) and the pair $C \in \mathbb{R}^{n \times n_s}$, $c \in \mathbb{R}^n_{>0}$ be such that the set $\mathcal{S} \triangleq \{x \in \mathbb{R}^{n_x} : Cx \leq c\}$ contains the origin in its interior. Then, given $\delta \in \mathbb{R}_{\geq 0}$, a relaxed weight recentered logarithmic barrier function for $\mathcal{S}$ is a function $\hat{B}_{\mathcal{S}} : \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{\infty\} : x \to \hat{B}_{\mathcal{S}}(x;\delta)$, where*

$$\hat{B}_{\mathcal{S}}(x;\delta) \triangleq \sum_{i=1}^{n} (1 + w_i) \left(\hat{B}\left(c_{(i)} - C_{(i)}x;\delta\right) + \ln(c_{(i)}) \right).$$

Moreover, weights $w_1, \ldots, w_n \in \mathbb{R}_{\geq 0}$ are such that the gradient of $f(x) \triangleq \hat{B}_{\mathcal{S}}(x;0)$ at the origin is null.

See [11] for details on the selection of the weights w_i.

II. Preliminaries

A. Problem setting

We consider a discrete-time linear system described by:

$$x_{k+1} = Ax_k + Bu_k, \tag{3}$$

where the index $k \in \mathbb{Z}_{\geq 0}$ denotes the time instant, $x_k \in \mathbb{R}^{n_x}$ is the state vector and $u_k \in \mathbb{R}^{n_u}$ is the applied input. Moreover, the system is subject to the following point-wise in time polytopic constraints,

$$x_k \in \mathcal{X} \triangleq \{x \in \mathbb{R}^{n_x} : A^x x \leq b^x\}, \tag{4a}$$

$$u_k \in \mathcal{U} \triangleq \{u \in \mathbb{R}^{n_u} : A^u u \leq b^u\}. \tag{4b}$$

The sequence $\boldsymbol{\xi}_{N+1}(\boldsymbol{u}_N, x^0)$ denotes the trajectory of system (3) starting from the initial condition $x^0 \in \mathbb{R}^{n_x}$ with control inputs taken from the sequence $\boldsymbol{u}_N \subset \mathbb{R}^{n_u}$.

In the following we study an MPC policy whose stability is guaranteed by selecting a terminal cost and terminal set constraint that satisfy the usual assumptions required for stability [1]. We consider a linear terminal control law with gain $K \in \mathbb{R}^{n_u \times n_x}$ and, for convenience, we define

$$A_K \triangleq A + BK, \text{ and, } \mathcal{X}_K \triangleq \{x \in \mathcal{X} : Kx \in \mathcal{U}\}.$$

We make the following assumption regarding the terminal set, inspired from [7].

Assumption 1. *The terminal set, $\mathcal{X}_f$, has the form*

$$\mathcal{X}_f \triangleq \{x \in \mathbb{R}^{n_x} : \psi(x) \leq 1\},$$

where $\psi : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ is convex, positive definite, continuously differentiable and satisfies the following contractive property:

$$\psi(A_K x) \leq \psi(x), \ \forall x \in \mathcal{X}_f.$$

Moreover, $\psi(\cdot)$ is positively homogeneous of degree 1, i.e.,

$$\forall \alpha \in \mathbb{R}_{\geq 0} : \psi(\alpha x) = \alpha \psi(x), \forall x \in \mathbb{R}^{n_x}.$$

A set such that Assumption 1 holds can be generated as an approximation to a polytopic set through the Minkowski functional [12]. Under Assumption 1 we have that $\forall \gamma \in \mathbb{R}_{\geq 0}$,

$$\gamma \mathcal{X}_f = \{x \in \mathbb{R}^{n_x} : \psi(x) \leq \gamma\}. \tag{5}$$

We now introduce an assumption on the constraint sets.

Assumption 2. *The sets $\mathcal{X}$, $\mathcal{U}$, $\mathcal{X}_f$ contain the origin in their interior. Moreover, $\mathcal{U}$ is bounded and $\mathcal{X}_f$ is compact.*

The N-step reachable set to the terminal set $\mathcal{X}_f \subseteq \mathbb{R}^{n_x}$ is defined as $\mathcal{R}_N(\mathcal{X},\mathcal{U},\mathcal{X}_f) \triangleq \{x^0 \in \mathbb{R}^{n_x} : \exists \boldsymbol{u}_N \subset \mathcal{U}$ such that $\xi_N(\boldsymbol{u}_N, x^0) \in \mathcal{X}_f$, and $\boldsymbol{\xi}_N(\boldsymbol{u}_N, x^0) \subset \mathcal{X}\}$. We use the short hand notation $\mathcal{R}_N(\mathcal{U},\mathcal{X}_f)$ to denote $\mathcal{R}_N(\mathbb{R}^{n_x},\mathcal{U},\mathcal{X}_f)$, and when clear from context we may omit its arguments.

B. Optimal control problem

The MPC controller we develop here is based on the OCP $\hat{\mathcal{P}}(x)$, defined as

$$\min_{\boldsymbol{u}_N} \hat{J}_N(\boldsymbol{u}_N, x) \triangleq \sum_{i=0}^{N-1} \hat{\ell}(u_i, \xi_i) + \hat{F}(\xi_N) \tag{6a}$$

$$\text{subject to } u_i \in \mathcal{U}, \ \forall i \in \mathbb{Z}_{[0,N-1]}, \ \xi_0 = x, \tag{6b}$$

$$\xi_{i+1} = A\xi_i + Bu_i, \ \forall i \in \mathbb{Z}_{[0,N-1]}. \tag{6c}$$

The stage cost, $\hat{\ell}$ and terminal cost, $\hat{F}$, are defined as

$$\hat{\ell}(x,u) \triangleq \ell(x,u) + \lambda_x \hat{B}_x(x;\delta_x), \ \lambda_x \in \mathbb{R}_{>0}, \tag{7a}$$

$$\hat{F}(x) \triangleq F(x) + \lambda_f \hat{B}_f(x;\delta_f), \ \lambda_f \in \mathbb{R}_{>0}, \tag{7b}$$

where $\ell(x,u) = \|x\|_Q^2 + \|u\|_R^2$, $F(x) = \|x\|_P^2$ with $Q \succeq 0$, $R \succ 0$ and $P \succ 0$. The functions $\hat{B}_x$, $\hat{B}_f$ are introduced to account for the state and terminal state constraints, which do not appear explicitly in constraints of $\hat{\mathcal{P}}(x)$. They are relaxed logarithmic barrier functions for the sets $\mathcal{X}$ and $\mathcal{X}_f$ with parameters δ_x, $\delta_f \in (0,1]$ and weights λ_x, $\lambda_f \in \mathbb{R}_{\geq 0}$. More precisely, the function $\hat{B}_f$ is a relaxed logarithmic barrier function for $\mathcal{X}_f$ (Definition 1):

$$\hat{B}_f(x;\delta_f) = \hat{B}(1 - \psi(x);\delta_f). \tag{8}$$

Similarly, $\hat{B}_x(x;\delta_x)$ is a relaxed weight-recentered logarithmic barrier function for the polytopic set $\mathcal{X}$. It is defined following Definition 2 for the pair $(A^{\mathcal{X}}, b^{\mathcal{X}})$. The following assumption ensures that the recentering of $\hat{B}_x(x;0)$ is preserved despite the relaxation.

Assumption 3. *The relaxation parameter $\delta_x \in (0,1]$ is chosen such that $\hat{B}_x(0;\delta_x) = 0$, $\nabla \hat{B}_x(0;\delta_x) = 0$.*

Such a relaxation parameter always exists. In particular, if $0 < \delta \leq \min(b^{\mathcal{X}})$, the recentering is preserved [7]. In the sequel, we let $B_x(x) \triangleq \hat{B}_x(x;0)$ and $B_f \triangleq \hat{B}_f(x;0)$, i.e., they denote the logarithmic barrier functions (before relaxation) that generate $\hat{B}_x$ and $\hat{B}_f$, respectively.

The OCP defined in (6) is a strongly convex optimization problem and Assumption 2 ensures the problem is feasible and has a unique global solution. We therefore define the minimizer function for $\hat{\mathcal{P}}_N(x)$ and associated state trajectory as $\hat{\boldsymbol{u}}_N^\star(x)$ and $\hat{\boldsymbol{\xi}}_{N+1}^\star(x)$, respectively. Moreover, we let $\hat{J}_N^\star(x)$ denote the optimal value.

The following assumption is sufficient to ensure stability and constraint satisfaction for any initial condition in the interior of $\mathcal{R}_N(\mathcal{X},\mathcal{U},\mathcal{X}_f)$ when considering MPC based on the unrelaxed variant of $\hat{\mathcal{P}}(x)$ where $\hat{B}_x$ and $\hat{B}_f$ are replaced by the unrelaxed barrier functions B_x and B_f. The result and proof follows [7, Theorem 1] and is omitted here for brevity.

Assumption 4. 4.1 *The matrix A_K is Schur.*

4.2 *There exists $M \in \mathbb{R}^{n_x \times n_x}$, $M \succ 0$ such that $B_x(x) \leq x^\top M x$, $\forall x \in \mathcal{N}$, where $\mathcal{N} \subset \mathcal{X}_K$, $0 \in \mathcal{N}^\circ$ and $\mathcal{N}$ is a convex and compact set.*

4.3 *The terminal cost matrix $P \in \mathbb{R}^{n_x \times n_x}$ is a solution to*

$$P = A_K^\top P A_K + K^\top R K + Q + \lambda_x M.$$

4.4 *The terminal set is contained in $\mathcal{N}$, i.e., $\mathcal{X}_f \subseteq \mathcal{N} \subset \mathcal{X}_K$.*

4.5 *The function B_f is a recentered barrier function for $\mathcal{X}_f$, satisfying $B_f(A_K x) - B_f(x) \leq 0$, $\forall x \in \mathcal{X}_f^\circ$.*

As stated in the introduction, one of the motivations for softening the state and terminal set constraints is to ensure the controller generates an input command even when state constraint violation is unavoidable. To this end, we derive a region of attraction (RoA) for which the closed-loop system under the MPC associated with $\hat{\mathcal{P}}(x)$ is asymptotically stable despite possible violation of the state constraints. The derivation of the RoA is detailed in the following section.

We note that one can also prove existence of an invariant subset of $\mathcal{R}_N(\mathcal{X},\mathcal{U},\mathcal{X}_f)$ for which the solution of $\hat{\mathcal{P}}(x)$ satisfies the state and terminal constraints, which also ensures stability of the MPC policy by [7, Theorem 1]. This subset can be made to approximate the reachable set arbitrarily well through selection of the relaxation parameters. The proof of the two aforementioned constraint satisfaction properties follow those of Theorem 2, Lemma 2, and Corollary 1 in [7] and have therefore been omitted here.

III. Theoretical analysis

A. Stability guarantees despite constraint violation

We now derive an RoA estimate for our MPC scheme leveraging the stability guarantees for MPC formulations without terminal set constraints detailed in [13]. The results in [13] require that the terminal set be a sublevel set of the terminal cost. However, for the OCP (6), a terminal set $\mathcal{X}_f$ respecting Assumption 1 is not necessarily a sublevel set of our terminal cost $\hat{F}$. As such, we first construct a fictitious terminal set, $\hat{\mathcal{X}}_f \subseteq \mathcal{X}_f$, that has the required properties for [13, Theorem 1] to hold and that can be made to approximate $\mathcal{X}_f$ arbitrarily well. More precisely, we let

$$\hat{\mathcal{X}}_f(\lambda_f,\delta_f) = \{x \in \mathbb{R}^{n_x} : \hat{F}(x;\delta_f) \leq \lambda_f \beta_f(0;\delta_f) + \hat{\alpha}\},$$

where $\hat{\alpha}$ is largest P-level set inside $\mathcal{X}_f$:

$$\hat{\alpha} = \max_{a \in \mathbb{R}} a \text{ such that } \mathcal{X}_P(a) \triangleq \{x : \|x\|_P^2 \leq a\} \subseteq \mathcal{X}_f. \quad (9)$$

The following result describes properties of $\hat{\mathcal{X}}_f$.

Lemma 1. *Under Assumptions 1, 2 and 4, the following identities hold:*

i) $x \in \mathcal{X}_f \iff \hat{B}_f(x;\delta) \leq \beta_f(0;\delta)$.

ii) $\mathcal{X}_P(\hat{\alpha}) \subseteq \hat{\mathcal{X}}_f \subseteq \mathcal{X}_f$.

iii) $x \in \hat{\mathcal{X}}_f \implies A_K x \in \hat{\mathcal{X}}_f$, *moreover*

$$\hat{F}(A_K x) - \hat{F}(x) + \hat{\ell}(x, Kx) \leq 0, \forall x \in \hat{\mathcal{X}}_f. \quad (10)$$

Proof. i) holds by monotonicity and positive definiteness of $\hat{B}_f$ after noting that $x \in \partial\mathcal{X}_f \iff \psi(x) = 1 \iff \hat{B}_f(x;\delta) = \beta_f(0;\delta)$. For the first inclusion in ii), take $x \in \mathcal{X}_P$, by definition of $\hat{\alpha}$, $x \in \mathcal{X}_f$ by i) this implies $\hat{B}_f \leq \beta_f(0;\delta)$ and as such $\hat{F}(x) \leq \hat{\alpha} + \lambda_f \beta_{\mathcal{X}_f}(0;\delta) \implies x \in \hat{\mathcal{X}}_f$. For the second inclusion in ii), take any $x \in \hat{\mathcal{X}}_f$, and assume $x \notin \mathcal{X}_f$. By i), it then holds that $\hat{B}_f(x) > \beta_f(0;\delta)$. Combining this with $x \in \hat{\mathcal{X}}_f$ we conclude that $\|x\|_P^2 < \hat{\alpha}$ as such $x \in \mathcal{X}_P(\hat{\alpha}) \subseteq \mathcal{X}_f$, which is a contradiction. For iii), take any $x \in \hat{\mathcal{X}}_f$, then, $\hat{F}(A_K x) - \hat{F}(x) + \hat{\ell}(x,Kx) = \|A_K x\|_P^2 - \|x\|_P^2 + \hat{\ell}(x,Kx) + \hat{B}_f(A_K x) - \hat{B}_f(x) \leq \hat{B}_f(A_K x) - \hat{B}_f(x) \leq 0$, where the first equality is obtained from the definition of $\hat{F}$; the first inequality from the definition of P in Assumption 4.3 as well as the upper bound on $\hat{B}_x$ given in Assumption 4.2; the second inequality from the inclusion of $\hat{\mathcal{X}}_f$ in $\mathcal{X}_f$ and the decrease condition in Assumption 4.5. □

Next we show that by adequately choosing the weight λ_f, the terminal set $\hat{\mathcal{X}}_f$ can be made to contain the set $\gamma\mathcal{X}_f$ for any $\gamma \in [0,1)$.

Theorem 1. *Let Assumptions 1, 2 and 4 hold. For any $\gamma \in [0,1)$ there exists $\bar{\lambda} \in \mathbb{R}_{>0}$ such that for all $\lambda_f \geq \bar{\lambda}$:*

$$\gamma\mathcal{X}_f \subseteq \hat{\mathcal{X}}_f(\lambda_f,\delta_f).$$

Proof. Let $\bar{\alpha} \in \mathbb{R}_{>0}$ be such that $F(x) \leq \bar{\alpha}$, $\forall x \in \gamma\mathcal{X}_f$, i.e., an upper bound on the function $F(x)$ over $\gamma\mathcal{X}_f$. Continuity of F and compactness of $\mathcal{X}_f$ ensure a finite $\bar{\alpha}$ exists. Then, define

$$\bar{\lambda} = \begin{cases} 0, & \text{if } \bar{\alpha} \leq \hat{\alpha} \\ \frac{\bar{\alpha}-\hat{\alpha}}{\beta(0;\delta_f)-\hat{B}_f(1-\gamma;\delta_f)}, & \text{else} \end{cases} . \quad (11)$$

Consider any $x \in \gamma\mathcal{X}_f$ and any $\lambda_f \geq \bar{\lambda}$. We show that

$$0 \leq \hat{\alpha} + \lambda_f \beta(0;\delta_f) - \hat{F}(x).$$

Starting from the left hand side:

$$\begin{aligned} \hat{\alpha}+\lambda_f\beta(0;\delta_f)-\hat{F}(x) &\geq \hat{\alpha}+\lambda_f\beta(0;\delta_f)-\bar{\alpha}-\lambda_f\hat{B}_f(1-\gamma;\delta_f) \\ &\geq \hat{\alpha}-\bar{\alpha}+\bar{\lambda}(\beta(0;\delta_f)-\hat{B}_f(1-\gamma;\delta_f)) \geq 0. \end{aligned}$$

The first inequality above comes from the definition of $\hat{F}$, the upper bound $\bar{\alpha}$, property (5) as well as positive definiteness and strict monotonicity of $\hat{B}_f$. The second inequality comes from $\lambda_f \geq \bar{\lambda}$ and the fact that the term it multiplies is positive by the properties of $\hat{B}_f$. The last inequality comes from the definition of $\bar{\lambda}$. $\square$

Remark 1. *If $\mathcal{X}_f$ is obtained as an inner approximation of a polytope, one can determine $\bar{\alpha}$ by either solving a constrained convex program or one may pick the highest value of $F(\cdot)$ evaluated at the vertices of the scaled polytope that generated $\mathcal{X}_f$, thanks to Bauer's maximum principle.*

It is worth noting that when $\nabla\psi(x) \neq 0$, for all x such that $\psi(x) = 1$, Theorem 1 provides a way to approximate $\mathcal{X}_f$ arbitrarily well. The result hinges on the fact that the 1-level set of ψ is then equivalent to the boundary of $\mathcal{X}_f$. This property holds for the Minkowski function approximation of a polytope discussed in [12].

We are now ready to state our main stability result which depends on a a lower bound on the value of the stage cost outside the set $\hat{\mathcal{X}}_f$, introduced hereunder.

Assumption 5. *Let d be a positive constant such that $\|x\|_Q^2 + \lambda_x \hat{B}_x(x;\delta_x) \geq d$ for all $x \notin \hat{\mathcal{X}}_f$.*

Theorem 2. *Let Assumptions 1-5 hold. Then, system (3) with $u_k = \hat{u}_0^\star(x_k) \in \mathcal{U}$ is asymptotically stable at the origin with a region of attraction $\hat{\Gamma}_N \subset \mathcal{R}_N(\mathcal{U}, \mathcal{X}_f)$, where*

$$\hat{\Gamma}_N = \left\{ x \in \mathbb{R}^{n_x} \;\middle|\; \begin{array}{r} J_N^\star(x) \leq \ell(x,0) + (N-1)d \\ +\lambda_f \beta_f(0;\delta) + \hat{\alpha} \end{array} \right\}. \quad (12)$$

Proof. The claim follows from [13, Theorem 1]. The result holds by the fact that: i) the terminal cost is a class $\mathcal{K}$-function by positive definiteness of P, positive definiteness of B_f, and $\lambda_f \in \mathbb{R}_{\geq 0}$; ii) the descent condition in (10) holds; iii) the terminal set is a sublevel set of the terminal cost; iv) for the OCP $\hat{\mathcal{P}}(\cdot)$, Assumption 5 implies that $\hat{\ell}(x,u) \geq d$ for all $x \notin \mathcal{X}_f$; and finally v) positive definiteness of $\hat{B}_x$ ensures positive definiteness of $\hat{\ell}$. $\square$

Remark 2. *If $\hat{\Gamma}_N \supset \mathcal{R}_N(\mathcal{X},\mathcal{U},\mathcal{X}_f)$, then the MPC based on $\hat{\mathcal{P}}(x)$ results in a larger RoA than the corresponding MPC with hard constraints. Furthermore, the closed-loop system is nominally robustly stable within $\hat{\Gamma}_N$ for sufficiently small additive disturbances. We emphasize that $\hat{\Gamma}_N$ is an inner-approximation of the true closed-loop RoA and that, in practice, the globally feasible MPC policy may (robustly) stabilize states outside of $\hat{\Gamma}_N$, or even $\mathcal{R}_N(\mathcal{U},\mathcal{X}_f)$.*

Next, we demonstrate that the relaxation parameters can be selected so that $\hat{\Gamma}_N$ approximates the interior of the reachable set $\mathcal{R}_N(\mathcal{U},\mathcal{X}_f)$ arbitrarily well. Before we state the result, we need to characterize the reachable set to $\hat{\mathcal{X}}_f$. Note that if $\mathcal{U}$ and $\mathcal{X}_f$ are compact, then $\mathcal{R}_N(\mathcal{U},\hat{\mathcal{X}}_f)$ is compact for all values of N. Secondly, it directly holds that $\forall \mathcal{A} \subseteq \mathcal{B} \subset \mathbb{R}^n$: $\mathcal{R}_N^\circ(\mathcal{U},\mathcal{A}) \subset \mathcal{R}_N^\circ(\mathcal{U},\mathcal{B})$. Finally, if $x \in \mathcal{R}_N^\circ(\mathcal{U},\mathcal{B})$ and $\boldsymbol{u}_N$ is such that $\xi_N(x,\boldsymbol{u}_N) \in \mathcal{B}$, then $\xi_1(x,\boldsymbol{u}_N) \in \mathcal{R}_{N-1}^\circ(\mathcal{U},\mathcal{B})$.

Let us now define an upper bound on the value of the stage cost over the i-step reachable set, with $i \in \mathbb{Z}_{\geq 0}$:

$$\mathcal{L}_i(\mathcal{U},\mathcal{X}) = \max_x \hat{\ell}(x,0), \text{ subject to } x \in \mathcal{R}_i^\circ(\mathcal{U},\mathcal{X}). \quad (13)$$

Theorem 3. *Let Assumption 1, 2 and 5 hold and assume $\mathcal{U}$ is compact. Take any $\rho \in \mathbb{R}_{[0,1)}$ and let $\bar{\delta}$ be such that*

$$\beta(0;\bar{\delta}) = \frac{N \max_{u\in\mathcal{U}} \|u\|_R^2 + \sum_{i=1}^{N-1} \mathcal{L}_i}{\lambda_f(1-\rho)} - \frac{\hat{\alpha}}{\lambda_f}, \quad (14)$$

where $\mathcal{L}_i$ is defined in (13), d is defined in Assumption 5, and $\hat{\alpha}$ is defined according to (9). Then, for all $x \in \mathcal{R}_N^\circ(\mathcal{U},\rho\hat{\mathcal{X}}_f)$, it holds that for all $0 < \delta_f \leq \bar{\delta}$, $x \in \hat{\Gamma}_N(\lambda_f;\delta)$.

Proof. Take any $x \in \mathcal{R}_N^\circ(\mathcal{U},\rho\hat{\mathcal{X}}_f)$. Then, there is a feasible input sequence $\boldsymbol{u}_N$ such that $\xi_N(x,\boldsymbol{u}_N) \in \rho\hat{\mathcal{X}}_f$. For the remainder of this proof we let $\boldsymbol{\xi}_N = \boldsymbol{\xi}_N(x,\boldsymbol{u}_N)$. By definition of the value function for $\hat{\mathcal{P}}_N(x)$ we have that

$$\begin{aligned} \hat{J}_N^\star(x) &\leq \hat{J}_N(\boldsymbol{u}_N,x) = \sum_{i=0}^{N-1} \hat{\ell}(\xi_i,u_i) + \lambda_f \hat{B}_f(\xi_N) \\ &\leq \hat{\ell}(x,0) + N \max_{u\in\mathcal{U}} \|u\|_R^2 + \sum_{i=1}^{N-1} \mathcal{L}_i + \lambda_f \hat{B}_f(\xi_N) \\ &\leq \hat{\ell}(x,0) + N \max_{u\in\mathcal{U}} \|u\|_R^2 + \sum_{i=1}^{N-1} \mathcal{L}_i + \rho(\hat{\alpha} + \lambda_f \beta(0;\delta_f)). \end{aligned}$$

Given the above inequalities, to ensure $x \in \hat{\Gamma}_N(\delta_f)$ it is sufficient to ensure

$$\begin{aligned} \hat{\ell}(x,0) + N \max_{u\in\mathcal{U}} \|u\|_R^2 + \sum_{i=1}^{N-1} \mathcal{L}_i + \rho(\hat{\alpha} + \lambda_f \beta(0;\delta_f)) \\ \leq \hat{\ell}(x,0) + (N-1)d + (\hat{\alpha} + \lambda_f \beta(0;\delta_f)). \end{aligned}$$

Rearranging the terms above we equivalently need to show that

$$\frac{N \max_{u\in\mathcal{U}} \|u\|_R^2 + \sum_{i=1}^{N-1} \mathcal{L}_i}{1-\rho} - \hat{\alpha} \leq \lambda_f \beta(0;\delta_f) + (N-1)d,$$

where the left hand side is equal to $\beta(0;\bar{\delta})$ by (14). The desired result is obtained by the assumption that $0 < \delta_f \leq \bar{\delta}$ and the fact that $\beta(0;\delta_f) \geq \beta(0;\bar{\delta})$. $\square$

The value of $\bar{\delta}$ satisfying (14) can be calculated by rearranging the definition in (1).

Remark 3. *Theorem 1 and Theorem 3 provide parameter design choices to ensure that $\hat{\mathcal{X}}_f$ approximates $\mathcal{X}_f$ and that $\hat{\Gamma}_N$ approximates $\mathcal{R}_N(\mathcal{U},\hat{\mathcal{X}}_f)$ arbitrarily well. Both results work in synergy: decreasing δ_f leads to a lowering of the lower bound on λ_f and increasing λ_f leads to an increase of the upper bound on δ_f.*

Remark 4. *Equation (14) requires a bound on the sum of the stage costs over any trajectory that starts in $\mathcal{R}_{N-1}(\mathcal{U},\mathcal{X}_f)$. The bound (13) corresponds to maximizing the value of the stage cost over each $\mathcal{R}_i$ for $i = 1,\ldots,N-1$, which might* *be conservative. In the case where A in* (3) *is invertible, the following problem is well posed and gives the tightest bound:*

$$\max_{\boldsymbol{u}_N} \sum_{i=1}^{N-1} \hat{\ell}(x_i, 0) \text{ subject to } \boldsymbol{u}_n \subset \mathcal{U},\ x_0 \in \hat{\mathcal{X}}_f,$$
$$x_{i+1} = A^{-1}x_i - A^{-1}Bu,\ i = 0, \dots, N-2.$$

Replacing $\hat{\mathcal{X}}_f$ by the polytope defining $\mathcal{X}_f$ we get the maximization of a convex function over a polytopic set which can readily be evaluated through Bauer's maximum principle.

Due to the very nature of the OCP formulation considered here, constraint violation may occur. As discussed in Section II, it is possible to derive bounds on the maximum constraint violation and a subset of $\hat{\Gamma}_N$ for which the closed-loop trajectories satisfy the constraints. These properties follow analogously to Lemma 4 and Theorem 6 in [7]

B. Optimization algorithms and suboptimal MPC

A key advantage of the proposed MPC formulation is that, assuming that $\mathcal{U}$ is simple, the OCP can be solved efficiently using projection-based methods such as the projected Newton method (PNM) [9]. Moreover, the latter being a descent method, it provides a suitable framework for analyzing the impact of a finite number of iterations of the optimizer.

In the reminder of this section we analyze the use of suboptimal solutions to the OCP (6) for input generation. We assume that the OCP is solved using an iterative solver with iterations defined by $\mathcal{I} : \mathbb{R}^{Nn_u} \times \mathbb{R}^{n_x} \to \mathbb{R}^{Nn_u}$, where

$$\mathcal{I}_i(\boldsymbol{u}^{(k)}, x) = \mathcal{I}(\mathcal{I}_{i-1}(\boldsymbol{u}^{(k)}, x), x),\ \text{ for all } i \in \mathbb{Z}_{\geq 1},$$
$$\mathcal{I}_0(\boldsymbol{u}^{(k)}, x) = \mathcal{J}(\boldsymbol{u}^{(k)}, x),$$

where i represents the number of iterations, x is the the OCP parameter, and $\boldsymbol{u}^{(k)}$ is a suitably defined initialization, e.g., the solution to the previous instance of the OCP. The mapping $\mathcal{J}$ provides the following warm-starting procedure:

$$\mathcal{J}(\boldsymbol{u}, x) = \boldsymbol{u}^+,\ \text{ where } u_j^+ = \begin{cases} u_{j+1}, & j < N \\ K\xi_N(x, \boldsymbol{u}), & j = N \end{cases}$$

We make the following assumption on the feasibility and contractivity of the optimizer, which holds for the PNM [9].

Assumption 6. *For any $\boldsymbol{u} \subset \mathbb{R}^{Nn_u}$ and $x \in \mathbb{R}^{n_x}$, the iteration $\mathcal{I}$ produces a feasible input, $\mathcal{I}(\boldsymbol{u}, x) \in \mathcal{U}$. Moreover,*

$$\hat{J}_N(\mathcal{I}(\boldsymbol{u}_N, x), x) - \hat{J}_N(\boldsymbol{u}_N, x) \leq 0.$$

We introduce a suboptimal trajectory of (3), defined as

$$\hat{\boldsymbol{u}}_N^{(k)}(\boldsymbol{i}, x^0) = \mathcal{I}_{i_k}(\hat{\boldsymbol{u}}_N^{(k-1)}, \hat{x}_k), \tag{15a}$$
$$\hat{x}_{k+1}(\boldsymbol{i}, x^0) = A\hat{x}_k + B\hat{u}_0^{(k)}, \tag{15b}$$

where the k^{th} element of the infinite sequence $\boldsymbol{i} \triangleq \{i_k\}_{k=0}^{\infty}$, i.e., $i_k \in \mathbb{Z}_{\geq 0}$, is the number of optimizer iterations performed at time instant k, $\hat{x}_0 = x^0$ and $\hat{\boldsymbol{u}}_N^{(-1)} = \boldsymbol{0}$. Theorem 4 gives asymptotic stability conditions for the closed loop system given in (15). We first derive a sufficient condition ensuring that trajectories of (3) end in the terminal set $\hat{\mathcal{X}}_f$.

Lemma 2. *Let Assumptions 1-5 hold and assume that the weights λ_f, δ_f have been selected so that* (14) *holds for some $\rho \in \mathbb{R}_{[0,1)}$. Then, for all pairs $(\boldsymbol{u}_N, x)$ such that*

$$\hat{J}_N(\boldsymbol{u}_N, x) \leq \hat{\ell}(x, 0) + (N-1)d + \rho(\hat{\alpha} + \lambda_f \beta(0; \delta_f)), \tag{16}$$

the associated N-step forward propagation of the state lies inside the terminal set, i.e.,

$$\xi_N(x, \boldsymbol{u}_N) \in \hat{\mathcal{X}}_f.$$

Proof. If $\boldsymbol{u}_N = \boldsymbol{0}$ and $\{\xi_i\}_{i=1}^{N-1} = \boldsymbol{0}$, the result holds through Assumption 2, so assume it is not so. We now show the result holds by contradiction. Let $x, \boldsymbol{u}_N$ be such a pair. Using the definition of $\hat{J}_N$ we have that $\sum_{i=0}^{N-1} \hat{\ell}(\xi_i, u_i) + \hat{F}(\xi_N) \leq \hat{\ell}(x, 0) + (N-1)d + \rho(\hat{\alpha} + \lambda_f \beta(0; \delta_f))$. Reordering and canceling the term $\hat{\ell}(x, 0)$ from both sides results in $\sum_{i=1}^{N-1} \hat{\ell}(\xi_i, u_i) + \|u_0\|_R^2 \leq (N-1)d + \rho(\hat{\alpha} + \lambda_f \beta(0; \delta_f)) - \hat{F}(\xi_N)$. The left hand side in the previous inequality can be strictly lower bounded by 0 as at least $\boldsymbol{\xi}_N$ or $\boldsymbol{u}_N$ are non zero and the stage cost is a positive definite function. Leveraging the assumption $\xi_N(x, \boldsymbol{u}_N) \notin \hat{\mathcal{X}}_f$ and the definition of $\hat{\mathcal{X}}_f$ we have the necessary condition: $0 < (N-1)d + (\rho - 1)(\hat{\alpha} + \lambda_f \beta(0; \delta_f))$. Rearranging (14), and letting $\bar{\delta} = \delta_f$ an equivalent condition for the previous inequality to hold is $0 < -(N \max_{u \in \mathcal{U}} \|u\|_R^2 + \sum_{i=1}^{N-1} \mathcal{L}_i - (N-1)d)$. It remains to show the right hand side of the last inequality is strictly negative. Recall the definition of d in Assumption 5 and that of $\mathcal{L}_i$ in (13) and note that $\mathcal{R}_i^\circ(\mathcal{U}, \hat{\mathcal{X}}_f) \supseteq \hat{\mathcal{X}}_f$ by forward invariance of $\hat{\mathcal{X}}_f$ (Lemma 1.iii). This ensures last two terms are at most equal. Assumption 2 and $R \succ 0$ ensure the first term is strictly negative, concluding the proof. ☐

Theorem 4. *Consider the trajectory $\hat{\boldsymbol{x}} \triangleq \{\hat{x}_k(\boldsymbol{i}, x)\}_{k=0}^{\infty}$ defined in* (15). *Let Assumptions 1-6 hold and assume that i_0*

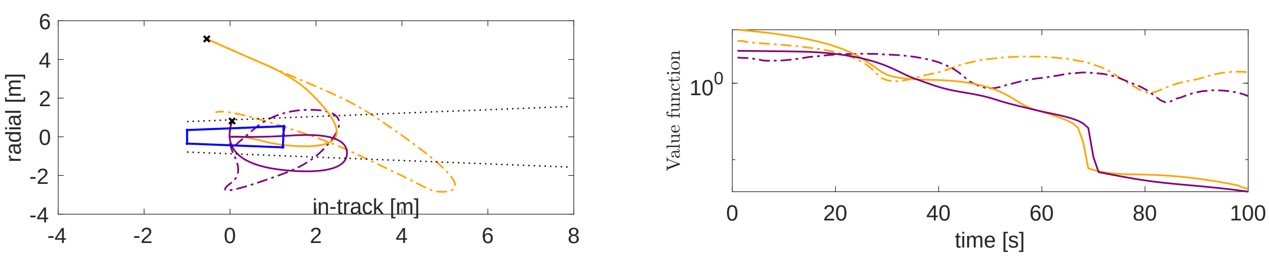

Fig. 1: Trajectories in the position plane (left), and evolution of the value function (right), for two initial conditions for the system controlled with controller 1 (solid lines) and with controller 2 (dashed lines). Cone constraint in black dotted lines.

is sufficiently large so that $\xi_N(x, \hat{\boldsymbol{u}}_N^0(i_0, x)) \in \rho\hat{\mathcal{X}}_f$*, for some* $\rho \in [0, 1)$*. Then, the closed loop trajectory,* $\hat{\boldsymbol{x}}$*, asymptotically converges to the origin for any sequence* $\{i_k\}_{k=1}^{\infty} \subseteq \mathbb{Z}_{\geq 0}$*.*

Proof. We first note that by Lemma 1.iii) for any $x, \boldsymbol{u}_N$ such that $\xi_N(\boldsymbol{u}_N, x) \in \hat{\mathcal{X}}_f$ it holds that

$$J(\xi_1, \mathcal{J}(\boldsymbol{u}_N, x)) - J(\boldsymbol{u}_N, x) \leq \hat{\ell}(x, u_0). \quad (17)$$

This implies that if $x, \boldsymbol{u}_N$ respect (16) then ξ_1 and the warm-started input sequence also respect (16). The contractive property in Assumption 6 ensures that this condition is still respected after any number of optimizer iterations. Thanks to this and noting that by assumption $\xi_N(\hat{x}_0, \hat{\boldsymbol{u}}^{(0)}) \in \hat{\rho}\mathcal{X}_f$, we conclude that at all time instants $k \in \mathbb{Z} > 0$, $\xi_N(\hat{x}_k, \hat{\boldsymbol{u}}^{(k)}) \in \hat{\mathcal{X}}_f$ by Lemma 2. The warm-starting procedure that occurs at each time instant ensures, through (17), that we have a strict decrease condition for the positive definite function $\hat{J}_N$ along the trajectory of the system, which ensures the result holds. □

IV. Numerical simulations

In this section we aim to illustrate the difference between our approach and the relaxed barrier MPC [7] for systems with hard input constraints. More precisely, we compare the difference between having hard input constraints in the OCP and saturating the output of the OCP in [7]. We consider a rendezvous maneuver of a Deputy spacecraft to a Chief spacecraft. The latter on a circular orbit, 500 km above the earth, and the motion of the Deputy restricted to the orbital plane of the Chief. The dynamics are approximated by the CWH equations [14], a set of linear equations with 4 states and 2 inputs. The components of the state vector represent the radial and in-track positions and velocities of the Deputy relative to the Chief resolved in the Chief centered Hill's frame. The input vector corresponds to applied accelerations along the radial and in-track directions.

During the maneuver, the Deputy is to reach an in-track relative position of 10 m with zero relative radial position. Moreover, the Deputy is required to maintain an in-track position of at least 8 m in front of the Chief, while remaining inside of a 5 degree cone with an apex at the Chief (Figure 1). Further, input saturation values of 0.01 N kg^{-1} and maximum relative velocity constraints of 3 m/s are considered.

The MPCs in this example use weights of $Q = \text{diag}(1, 10^{-2}, 10^{-1}, 10^{-2})$ and $R = 10^{-3}I$, a prediction horizon $N = 40$, a sampling period of 1 s, and a terminal gain K given by the solution to the DARE with parameters $Q = I$ and $R = 10^4 I$. The terminal cost matrix P is obtained following the procedure in [12, Lemma 1, (7)] with γ of [12, Lemma 1, (7)] given as $\frac{1}{d_{\mathtt{min}}^2} + 5 * 10^3$. A polytopic terminal set is obtained starting from the MOAS [15] for dynamics $x_+ = \frac{1}{0.97} A_K x$ and with scaling [12, Lemma 2]. We let $\lambda_x = 10^{-4}$, $\delta_x = 7 * 10^{-2}$ and chose $\lambda_f = 5 * 10^3$, which is the same order of magnitude as the entries of P. Finally, δ_f is selected so that (14) and (11) holds for $\rho = 0.97$. The controller obtained from solving (6) with the above parameters is denoted by *controller 1*. We compare this to an other controller, *controller 2*, with similar parameters and defined following [7], i.e. by relaxing all state and input constraints in the OCP and by applying saturation to the desired input.

One hundred initial conditions were generated pseudo randomly and scaled using bisection to be included in $\hat{\Gamma}_N$. To illustrate the difference in performance with the two approaches we consider reaching a distance of 0.05 m from the origin as an indicator of convergence. All trajectories obtained with controller 1 had converged after 50 s. In contrast, none of the trajectories for controller 2 had converged in 50 s. After 100 s, 30 trajectories had converged. Figure 1 (left) illustrates typical trajectories with controller 1 and 2. Figure 1 (right) shows the evolution of the value function for controller 1 and controller 2. As derived in theory, the value function for controller 1 is strictly monotonically decreasing in contrast to that of controller 2. This hints to a destabilizing effect caused by the external saturation. Indeed, when discarding the saturation a strict decrease was observed and is theoretically confirmed by the stability analysis of [7]. Note that a range of tuning parameters were tested for controller 2 to see if the adverse effect could be avoided, but all values resulted in similar behavior for a large set of initial conditions.

V. Conclusion

In this paper we studied the stability properties of an MPC scheme with hard input constraints and soft state constraints encoded through relaxed barrier functions. A region of attraction estimate was provided and stability results for a suboptimal implementation were derived. Future work will focus on the development of a sparsity-exploiting iteration scheme akin to [8], as well as studying the robust stability properties of the proposed scheme under bounded disturbances.

References

[1] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, no. 6, pp. 789–814, 2000.

[2] P. O. Scokaert and J. B. Rawlings, "Feasibility issues in linear model predictive control," *AIChE Journal*, vol. 45, no. 8, pp. 1649–1659, 1999.

[3] V. Gracia, P. Krupa, D. Limon, and T. Alamo, "Implementation of soft-constrained MPC for tracking using its semi-banded problem structure," *IEEE Control Systems Letters*, vol. 8, pp. 1499–1504, 2024.

[4] A. Zheng and M. Morari, "Stability of model predictive control with mixed constraints," *IEEE Transactions on Automatic Control*, vol. 40, no. 10, pp. 1818–1823, 1995.

[5] S. V. Raković, S. Zhang, H. Sun, and Y. Xia, "Model predictive control for linear systems under relaxed constraints," *IEEE Transactions on Automatic Control*, vol. 68, no. 1, pp. 369–376, 2021.

[6] M. N. Zeilinger, M. Morari, and C. N. Jones, "Soft constrained model predictive control with robust stability guarantees," *IEEE Transactions on Automatic Control*, vol. 59, no. 5, pp. 1190–1202, 2014.

[7] C. Feller and C. Ebenbauer, "Relaxed logarithmic barrier function based model predictive control of linear systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 3, pp. 1223–1238, 2016.

[8] ——, "Sparsity-exploiting anytime algorithms for model predictive control: A relaxed barrier approach," *IEEE Transactions on Control Systems Technology*, vol. 28, no. 2, pp. 425–435, 2018.

[9] D. P. Bertsekas, "Projected Newton methods for optimization problems with simple constraints," *SIAM Journal on control and Optimization*, vol. 20, no. 2, pp. 221–246, 1982.
[10] J. Hauser and A. Saccon, "A barrier function method for the optimization of trajectory functionals with constraints," in *Proceedings of the 45th IEEE Conference on Decision and Control*. IEEE, 2006, pp. 864–869.
[11] C. Feller and C. Ebenbauer, "Weight recentered barrier functions and smooth polytopic terminal set formulations for linear model predictive control," in *2015 American Control Conference (ACC)*. IEEE, 2015, pp. 1647–1652.
[12] ——, "Barrier function based linear model predictive control with polytopic terminal sets," in *53rd IEEE Conference on Decision and Control*. IEEE, 2014, pp. 6683–6688.
[13] D. Limon, T. Alamo, F. Salas, and E. F. Camacho, "On the stability of constrained MPC without terminal constraint," *IEEE Transactions on Automatic Control*, vol. 51, no. 5, pp. 832–836, 2006.
[14] W. Clohessy and R. Wiltshire, "Terminal guidance system for satellite rendezvous," *Journal of the aerospace sciences*, vol. 27, no. 9, pp. 653–658, 1960.
[15] E. G. Gilbert and K. T. Tan, "Linear systems with state and control constraints: The theory and application of maximal output admissible sets," *IEEE Transactions on Automatic Control*, vol. 36, no. 9, pp. 1008–1020, 2002.