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TR2024-091    July 09, 2024

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American Control Conference (ACC) 2024
A Switched Reference Governor for High Performance Trajectory Tracking under State and Input Constraints

Nan Wang¹, Stefano Di Cairano², and Ricardo G. Sanfelice³

Abstract—This paper proposes a switched reference governor (RG) algorithm to achieve rapid and non-oscillatory convergence to a given reference signal while satisfying the imposed constraints by switching between a fast and oscillatory controller and a slow and non-oscillatory controller. The switched RG first computes the set of the pairs of state and its admissible references for both controllers in an offline fashion. For the online computation, at each iteration, the proposed algorithm computes the admissible reference sets for each controller at the current state. Then, the algorithm activates one of the controllers based on the closeness between the system state and the reference. At last, a lightweight optimization problem to find the admissible reference that is closest to the reference signal is solved and the solution, which is referred to as virtual reference, is applied to the control system as the reference signal. Through measuring the closeness between the system state and the reference by a Lyapunov function and a discrete-time hybrid system model, we show robust switching, recursive feasibility and convergence of the virtual reference to the reference signal, among other key properties of the proposed switched RG.

I. INTRODUCTION

Enforcing constraints in control systems by design is a new challenge in many control applications. Constraints often manifest as actuator magnitude and rate limitations, allowed ranges of process variables for safe, efficient system function, and requirements for collision avoidance. There are now several impactful research outcomes on constraint-handling methodologies, such as model predictive control (MPC) [1], control barrier function (CBF) [2], [3], and reference governor (RG) [4]. Among these, MPC necessitates a comprehensive controller redesign incorporating constraints and it is known to result in significant computational burden. Similarly, CBF poses design challenges especially for recursive feasibility when both input and output constraints are present, and requires solving time-consuming optimization problem. RG is a state-feedback control law that modulates the reference signal of a pre-stabilized plant. Compared to MPC, RG utilizes existing or legacy controllers and supplements them with constraint management capability. Different from CBF, RG may not require optimization and systematic methods to achieve recursive feasibility for a unique class of smooth systems. Thus, RG leads to a streamlined design and enhanced computational efficiency, albeit at the cost of reduced closed-loop performance [5].

Switching control strategies have garnered significant attention for its potential to improve the performance degraded by RG. In [6], a switching RG mechanism for varying system operating points is introduced, alongside a proposed finite state machine for transitioning between RG paradigms. A framework presented in [7] utilizes remote control of nonlinear discrete-time systems and dynamically transitions between two feedback controllers based on remotely computed virtual reference commands. The approach in [8] employs a supervisory RG method for load/frequency control in networked multi-area power systems, enhancing disturbance rejection performance by switching between multiple RGs that consider different configurations of disturbance. Specialized schemes are also emerging, such as a switched RG designed specifically for linear motor-driven systems in [9] and a data-driven switched RG for constrained braking systems in [10], based on the methodology presented in [11]. However, most of the switching mechanism in those papers are designed from experience or driven by data, which do not allow to get theoretical analysis of their properties.

This paper presents a switched reference governor underpinned by Lyapunov function techniques that transitions between a fast and oscillatory controller and a slow but non-oscillatory controller. The controllers with both characteristics are commonly encountered in control system design and can be exemplified using a double-integrator system:

\[ x^+ = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \]

where \( x := (x_1, x_2) \in \mathbb{R}^2 \) and \( u \in \mathbb{R} \). Two controllers, denoted \( \gamma_1 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \) and \( \gamma_2 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \), are designed for [1] as follows: \( \gamma_1(x, v) := [-0.5 \quad -0.5] x + 0.5v \), \( \gamma_2(x, v) := [-0.25 \quad -1.5] x + 0.25v \) where \( v \in \mathbb{R} \) is the exogenous (virtual) reference. With the same reference, illustrated in Figure 1a, the state trajectory achieves very close proximity (1% error) to the actual reference roughly after 17 seconds when employing \( \gamma_1 \), classifying it as a fast controller. Conversely, when \( \gamma_2 \) is applied, it takes approximately 50 seconds for the state trajectory to reach very close proximity (1% error) to the actual reference for the first time, with no subsequent oscillations observed. Consequently, this controller is categorized as a slow controller. This behavior serves as motivation for the switched RG algorithm proposed in this paper, which leverages the key characteristics of each tracking controller while ensuring constraint satisfaction.

In this paper, we design a switched RG that combines controllers to effectively utilize their individual advantages, such as a fast convergence rate, or avoiding oscillations,
and ensures that the constraints are recursively satisfied. The proposed switched RG is designed by determining the respective sets of state-reference pairs for the closed-loop systems controlled by fast controller and slow controller, where the closed-loop system satisfies constraints. When the switched RG is running online, for each iteration the switched RG determines the sets of admissible reference for fast controller and slow controller at the current state. The decision to switch to an alternate controller is contingent on whether these reference sets are nonempty and also on the value of a Lyapunov function. After the selection of an appropriate controller, the proposed switched RG selects the virtual reference within the corresponding admissible reference set that has the minimal distance to the reference within the corresponding admissible reference set, which we refer to as the virtual reference. As in this paper, we propose an algorithm that switches between two controllers, $\gamma_1$ and $\gamma_2$, we denote the closed-loop system controlled by $\gamma_1$ as $P_1$ and the closed-loop system controlled by $\gamma_2$ as $P_2$. For any virtual reference $k \mapsto v(k)$, the solution to $P$ in (2) is a function $\phi : \mathbb{N} \to \mathbb{R}^n$ such that $\phi(k+1) = G(\phi(k), v(k))$ for each $k \in \text{dom} \phi = \text{dom} v$. In addition, a solution $\phi$ is complete if $\text{dom} \phi$ is unbounded, nontrivial if $\text{dom} \phi$ contains at least two points, and maximal if there does not exist another solution $\phi'$ such that $\text{dom} \phi \subset \text{dom} \phi'$ and $\phi(k) = \phi'(k)$ for each $k \in \text{dom} \phi$.

C. Stability, Attractivity, and Asymptotic Stability

For the analysis of properties, we provide definitions for stability, attractivity, and asymptotic stability. We first introduce a set-valued map $\mathcal{A}^P : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, which, given a virtual reference $v$, collects all the equilibria of the closed-loop system $P$ in (2). Mathematically,

$$v \mapsto \mathcal{A}^P(v) := \{x_p \in \mathbb{R}^n : E(x_p) = v, G(x_p, v) = x_p\}.$$

(3)

Definition 2.1 (Asymptotic stability): Given $v \in \mathbb{R}^m$ and the discrete-time closed-loop system $P$ as in (2) such that $\mathcal{A}^P(v) \subset \mathbb{R}^n$ is nonempty, the set $\mathcal{A}^P(v)$ is said to be

1) Lyapunov stable for $P$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that each solution $\phi$ to $P$ with $|\phi(0)|_{\mathcal{A}^P(v)} \leq \delta$ satisfies $|\phi(k)|_{\mathcal{A}^P(v)} \leq \epsilon$ for each $k \in \text{dom} \phi$.

2) Attractive for $P$ with basin of attraction $\mathcal{U} \subset \mathbb{R}^n$ if each maximal solution $\phi$ to $P$ with $\phi(0) \in \mathcal{U}$ is complete and satisfies $\lim_{t \to \infty} |\phi(k)|_{\mathcal{A}^P(v)} = 0$.

3) Asymptotically stable for $P$ with basin of attraction $\mathcal{U}$ if it is Lyapunov stable and attractive with basin of attraction $\mathcal{U}$.

For any virtual reference $v \in \mathbb{R}^m$, the set $\mathcal{A}^{P_1}(v)$ is assumed to be attractive for $P_1$ and $\mathcal{A}^{P_2}(v)$ is assumed to be asymptotically stable for $P_2$, as explicitly articulated later in Assumption 3.1. This facilitates the designing of a switching scheme between $P_1$ and $P_2$. 

Fig. 1: Double integrator example and the switched RG framework.
III. PROBLEM FORMULATION

In this paper, we design a switched RG for the closed-loop system (2) that is equipped with a fast controller and a slow controller to track the given reference while satisfying the constraints. The problem is formulated as follows.

**Problem 1:** Given two discrete-time closed-loop systems (2) obtained from controllers $\gamma_1$ and $\gamma_2$, develop a reference governor to switch between closed-loop systems $P_1$ and $P_2$, utilizing their distinct characteristics (specifically, slow convergence without overshoot and fast convergence with overshoot), while ensuring that the constrained output $y$ lies within the set $Y$.

The switched RG unites the fast controller, used globally to converge rapidly, with the slow controller, used when close to the reference to avoid oscillations. As is shown in Figure 1, the switched RG consists of two key modules: a switch module and an RG module. The switch module is responsible for switching between the fast controller and slow controller, while the RG module is designed to compute the virtual reference for the currently selected controller. We make the following assumption.

**Assumption 3.1:** Given reference $r \in \mathbb{R}^m$, for each $v \in \mathbb{R}^m$, both $A^{P_1}(v)$ and $A^{P_2}(v)$ are nonempty, and there exist set-valued maps $U_0 : \mathbb{R}^m \Rightarrow \mathbb{R}^n$ and $E : \mathbb{R}^m \Rightarrow \mathbb{R}^n$ such that $U_0(r)$ is an open set and contains an open neighborhood of $A^{P_1}(r)$, $E(r)$ is a closed set, and

1) $A^{P_1}(r)$ is a subset of $E(r)$ and for each $v \in \mathbb{R}^m$, $A^{P_1}(v)$ is attractive for $P_1$ with basin of attraction $\mathbb{R}^n$;
2) for each $v \in \mathbb{R}^m$, $A^{P_2}(v)$ is asymptotically stable for $P_2$, and, particularly, $A^{P_2}(r)$ is asymptotically stable for $P_2$ with basin of attraction containing $U_0(r)$;
3) there exist positive constants $\delta_0$ and $\delta_0$, and a set-valued map $T_{1,0} : \mathbb{R}^m \Rightarrow \mathbb{R}^n$ such that $T_{1,0}(r)$ is a closed set,

$$E(r) + \delta_0 B \subset T_{1,0}(r), \quad T_{1,0}(r) + 2\delta_0 B \subset U_0(r)$$

and each solution to the closed-loop system $P_2$ with initial condition in $T_{1,0}(r)$ resulting from applying $\gamma_2$ remains in $U_0(r)$.

**Remark 3.2:** Items 1 and 2 in Assumption 3.1 assert that the fast controller $\gamma_1$ induces global attractiveness to $A^{P_1}(v)$ for any $v$ produced by RG and, as RG drives $v$ to converge to $r$, eventually to $A^{P_1}(r)$ for $P_1$, which is situated within the basin of attraction of $A^{P_2}(r)$ for $P_2$. Item 3 in Assumption 3.1 guarantees the inclusion relationship between $E(r)$ and $U_0(r)$, such that the switching mechanism is able to unite $\gamma_1$ and $\gamma_2$ to solve Problem 1. The sets $U_0(r)$ and $E(r)$ may be constructed as sublevel sets of a Lyapunov function for the designed switched RG system, which is discussed later in Remark 5.3.

The proposed RG is designed to track a discrete-time exogenous desired reference $r$ for the performance output $z$. RG is formulated to compute the virtual reference $v$ derived based on the current state $x$ and the reference $r$, which is then applied to system (2). Consequently, by inputting the virtual reference $v$, instead of the reference $r$ into the closed-loop system (2), the constraint $y \in Y$ is maintained.

Most RG methodologies update the virtual reference, $v$, at every time instance. If $v$ is constantly applied from a particular time instant onward, the resulting output will always adhere to the imposed constraints. A maximal output admissible set $O^\infty_{v}$ comprises all states $x$ and constant virtual references $v$ such that the predicted response initiating from state $x$ and using a constant virtual reference $v$ satisfies the constraints at all future time instances, namely, $O^\infty_{v} := \{(x, v) \in \mathbb{R}^m \times \mathbb{R}^n : \dot{y}_P(k|x, v) \in Y \forall k \in \mathbb{N}_+\}$, where $\dot{y}_P(k|x, v)$ denotes the predicted constrained output $y$ of the closed-loop system $P$ at time $k$ when starting from the initial state $x$ with constant virtual reference $v$. The computation of $O^\infty_{v}$ is typically performed offline, requiring no calibration, and, in most cases, involves polynomial computational complexity. For nonlinear systems, it is more convenient to compute an invariant subset of $O^\infty_{v}$, denoted $S^\infty_{v}$ and defined as follows:

$$S^\infty_{v} := \{(x, v) \in S^P : (v, G(x, v)) \in S^\infty_{v}\}$$

where $S^P$ is some subset of $O^\infty_{v}$. Note that set $S^\infty_{v}$ is a subset of $O^\infty_{v}$, implying the admissibility of all points in $S^\infty_{v}$. For each $(x, v) \in S^\infty_{v}$, we have $(v, G(x, v)) \in S^\infty_{v}$, ensuring forward invariance. The following assumption posits that the pairs of reference $r$ and its equilibria are contained in both $S^\infty_{v}$ and $S^\infty_{r}$.

**Assumption 3.3:** Given reference $r \in \mathbb{R}^m$, for $i \in \{1, 2\}$, we have $(r, x_p) \in \mathbb{R}^m \times \mathbb{R}^n : x_p \in A^{P_i}(r) \subset S^\infty_{v}$.

IV. METHODOLOGY

The switched RG algorithm relies on the output admissible sets of $P_1$ and $P_2$, respectively, $S^\infty_{v}$ and $S^\infty_{r}$, as outlined in (4). The construction of these sets can be accomplished in an offline fashion using pre-existing methodology for computing invariant sets, including the iterative method, the optimization-based method, and the polyhedral method. In this paper, the polyhedral method is employed. The switched RG algorithm performs the following steps:

1) At the current state $x$, compute the admissible reference

$$V_q(x) = \{v \in \mathbb{R}^m : (v, x) \in S^\infty_{v}\} \quad \forall q \in \{1, 2\}. \quad (5)$$

2) Evaluate the switching logic and update the mode $q$ if necessary.
3) Solve the following optimization problem:

**Problem 2:** Given the reference $r \in \mathbb{R}^m$, the current mode $q \in \{1, 2\}$, and the current state $x \in \mathbb{R}^n$, solve

$$\arg\min_{v \in V_q(x)} |v - r|^2. \quad (6)$$

The optimization process for solving Problem 2 simply involves a single-dimensional search within the set $V_q(x)$, which can be performed numerically quite efficiently.
A. Switching Logic based on Lyapunov Function

We propose the following switching logic.

Switching Logic: Given the current state $x$, the current mode $q \in \{1, 2\}$, where $q = 1$ corresponds to the fast controller and $q = 2$ corresponds to the slow controller, and the reference $r$,

1) If $V_1(x)$ is empty and $V_{2-q}(x)$ is nonempty, then $q$ is reset to $3 - q$, making $P_{3-q}$ the active closed-loop system.

2) If $q = 1$, namely, the active closed-loop system is $P_1$, and
   a) both $V_1(x)$ and $V_2(x)$ are nonempty;
   b) the state $x$ is “close” to the reference $r$;
   c) the virtual reference provided for $P_2$ is closer to $r$ than the previous virtual reference, then it resets $q$ to 2, so that the active closed-loop system is $P_2$ due to its nonoscillatory performance.

3) If $q = 2$, namely, the active closed-loop system is $P_2$, and
   a) both $V_1(x)$ and $V_2(x)$ are nonempty;
   b) the state $x$ is “far away” from the reference $r$;
   c) the virtual reference provided for $P_1$ is closer to $r$ than the previous virtual reference, then it resets $q$ to 1, so that the active closed-loop system is $P_1$ due to its fast convergence rate.

In items 2.b and 3.b of the switching logic, the distance between the state $x \in \mathbb{R}^n$ and the reference $r \in \mathbb{R}^m$ needs to be properly defined. Note that $x$ and $r$ may have different dimensions, making the norm of $x - r$ not suitable as distance. In this paper, a Lyapunov function is employed to capture this “closeness”. To define the Lyapunov function properly, we start with defining the positive definite functions.

**Definition 4.1 (Positive definite function):** A function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is positive definite, also written $\rho \in PD$, if $\rho(s) > 0$ for all $s > 0$ and $\rho(0) = 0$.

With this definition in place, we define a Lyapunov function notion parameterized by the virtual reference $v$.

**Definition 4.2 (Lyapunov function):** Given the closed-loop system $P$ in (2), and the reference $r \in \mathbb{R}^m$, a function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is called a Lyapunov function for $P$ relative to $A^P(r)$ within the set $S \subset \mathbb{R}^n$ if

1) $V(x, r) > 0$ for all $x \in S \setminus A^P(r)$, and $V(x, r) = 0$ for all $x \in A^P(r)$,

2) $V(G(x, r), r) - V(x, r) \leq -\rho(|x| A^P(r))$ for all $x \in S$, where $\rho \in PD$ is continuous.

**Remark 4.3:** In Assumption B.1 $P_2$ is assumed to have the set $A^{P_2}(r)$ asymptotically stable with a basin of attraction containing some $U_0(r)$ for each $r \in \mathbb{R}^m$. By converse theorem in [12], there exists a Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ for $P_2$ relative to $A^{P_2}(r)$ within $U_0(r)$.

The SwitchingLogic for updating the mode is detailed in Algorithm 1. In addition to $x$, $q$, $S_{\infty}^{P_1}$, $S_{\infty}^{P_2}$ and $r$, we also employ $c_1$ and $c_2$ such that $c_1 > c_2$ as the threshold of the Lyapunov function to trigger the switch.

Setting $c_1 > c_2$ avoids chattering between the controllers. We also define $v_{prev}$ as the previous virtual reference. In Lines 3 - 7, Algorithm 1 initially handles the corner case in item 1 in Switching Logic where no admissible input is available. Item 2 in Switching Logic is implemented in Lines 8 - 9. Item 3 in Switching Logic is implemented in Lines 10 - 12. If no switch is triggered, the return mode is assigned as the current mode in Line 13.

B. A Switched RG Implementation

The proposed switched RG is summarized in Algorithm 2. The discrete time, state, and mode are initialized in Line 1. For each iteration, the current reference is updated in Line 2. The discrete time, state, and mode are initialized in Line 3. If no switch is triggered, the return mode is assigned as the current mode in Line 4. The current mode is then updated by a call to the SwitchingLogic function in Line 5. Using this updated mode, the virtual reference is computed by solving Problem 2 in Lines 6 - 7. The trajectory $k \mapsto x(k)$ is obtained by applying the computed virtual reference $v$ to the selected closed-loop system $P_q$ in Line 8. Data updates occur at the end of each iteration, setting the stage for the algorithm to progress to the next iteration.

V. Analysis of Theoretical Properties

This section introduces the hybrid system modeling of the proposed switched RG, following the theoretical guarantees.
A. Discrete-time Hybrid System Model

To formally represent the switched system, an auxiliary state variable \( q \in \{1, 2\} \) is introduced to distinguish between the two closed-loop systems under consideration. Additionally, \( v_{\text{prev}} \in \mathbb{R}^m \) is incorporated as a state variable to capture the virtual reference value at the previous time, namely, \( k - 1 \). The given reference \( r \in \mathbb{R}^m \) acts as a constant parameter in this model. In practice, if the given reference is piecewise constant and RG detects that the value of the reference changes, then the switched RG is restarted. By adding a logic variable \( q \), we obtain a hybrid system model in discrete time, which we formulate next. For \( q \in \{1, 2\} \), the RG function, defined as \( n_q : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m \), is such that given \( r \in \mathbb{R}^m \) and \( x \in \mathbb{R}^n \), \( n_q(x, r) \) equals the solution to \( \text{Problem 2} \) for \( r \) and \( \mathcal{V}_q(x) \). We define the extended state \( \chi := (x, q, v_{\text{prev}}) \in \mathbb{R}^n \times \{1, 2\} \times \mathbb{R}^m \). Given a constant \( r \in \mathbb{R}^m \), the switched RG system can be modeled as

\[
\mathcal{H}_s : \begin{cases}
\chi^+ = \begin{bmatrix} G_q(x, n_q(x, r)) \\ q \\ \kappa_q(x, r) \end{bmatrix} =: f_s(\chi, r) & (x, r) \in C_s \\
\chi^+ = \begin{bmatrix} x \\ 3 - q \\ v_{\text{prev}} \end{bmatrix} =: g_s(\chi, r) & (x, r) \in D_s
\end{cases}
\]

where

\[
D_s := D_{1} \cup D_{2} \quad \text{and} \quad C_s := (\mathbb{R}^n \times \{1, 2\} \times \mathbb{R}^m) \setminus (D_{s}) \quad \text{with} \quad D_{1} := D_{1}^{(1)} \cup D_{2}^{(1)}, \quad D_{2} := D_{1}^{(2)} \cup D_{2}^{(2)}, \quad \text{and} \quad D_{1}^{(1)} := \{ (x, q, v_{\text{prev}}, r) \in \mathbb{R}^n \times \{1, 2\} \times \mathbb{R}^m : V_1(x) \neq \emptyset \ \forall \ i \in \{1, 2\}, V_1(r) \leq c_2 q = 1, \inf_{x \in V_1(x)} |v - r| \leq |v_{\text{prev}} - r| \}, \\
D_{1}^{(2)} := \{ (x, q, v_{\text{prev}}, r) \in \mathbb{R}^n \times \{1, 2\} \times \mathbb{R}^m : V_1(x) = \emptyset, V_2(x) \neq \emptyset \}, \quad D_{2}^{(1)} := \{ (x, q, v_{\text{prev}}, r) \in \mathbb{R}^n \times \{1, 2\} \times \mathbb{R}^m : V_1(x) \neq \emptyset \ \forall \ i \in \{1, 2\}, V_1(r) \\
\geq c_2 q = 2, \inf_{x \in V_1(x)} |v - r| \leq |v_{\text{prev}} - r| \}, \\
D_{2}^{(2)} := \{ (x, q, v_{\text{prev}}, r) \in \mathbb{R}^n \times \{1, 2\} \times \mathbb{R}^m : V_2(x) = \emptyset, V_1(x) \neq \emptyset \}.
\]

Given the initial state \( x_0 \in \mathbb{R}^n \) and initial mode \( \phi_0 \in \{1, 2\} \), the initial extended state is set as \( \chi_0 = (x_0, q_0, v_{\text{prev}}) \). We denote the \( x, q, \) and \( v_{\text{prev}} \) component of a solution \( \phi \) to (7) as \( \phi_x, \phi_q, \) and \( \phi_{v_{\text{prev}}} \), respectively.

Remark 5.1: Note that \( C_s \cup D_s = \mathbb{R}^n \times \{1, 2\} \times \mathbb{R}^m \). Then, for any \( \chi \in \mathbb{R}^n \times \{1, 2\} \times \mathbb{R}^m \) and \( r \in \mathbb{R}^m \), we have \( f_s(\chi, r) \in C_s \cup D_s \) and \( g_s(\chi, r) \in C_s \cup D_s \). Therefore, for any initial \( \chi_0 \in \mathbb{R}^n \times \{1, 2\} \times \mathbb{R}^m \), a nontrivial solution to (7) starting from \( \chi_0 \) is guaranteed to exist, and every maximal solution to (7) is complete if \( k \rightarrow v(k) \) is complete.

Remark 5.2: Note that \( C_s \) and \( D_s \) are both closed sets. If \( G_q \) and \( n_q \) for \( q \in \{1, 2\} \) are both continuous functions, then \( f_s \) and \( g_s \) are both continuous. Therefore, hybrid system data \( (C_s, f_s, D_s, g_s) \) satisfies the regularity conditions, namely, hybrid basic conditions in [13].

Remark 5.3: Given \( r \in \mathbb{R}^m \), the sets \( U_0(r) \) and \( T_{1,0}(r) \) in Assumption 5.1 can be constructed as

\[
T_{1,0}(r) := \{ x \in \mathbb{R}^n : V(x, r) \leq c_1 \}.
\]

B. Recursive Feasibility

Recursive feasibility guarantees that if there exists a feasible control at the first time instance, then a feasible control will be found at all the following time instances. The following assumption posits that a feasible control solution can be derived given the initial condition.

Assumption 5.4: Given \( r \in \mathbb{R}^m \), the initial state \( x_0 \in \mathbb{R}^n \) and initial mode \( \phi_0 \in \{1, 2\} \) such that \( V_{\phi_0}(x_0) \) is nonempty.

Next, Theorem 5.5 ensures a feasible solution to Problem 2 at each time.

Theorem 5.5: (Recursive feasibility) Given a reference \( r \in \mathbb{R}^m \) and \( (x_0, q_0) \in \mathbb{R}^n \times \{1, 2\} \) satisfying Assumption 5.4, there exists a range of infinite horizon admissible reference \( \phi_{x}(k, j)(\phi_x(k, j)) \) is not an empty set for all \((k, j) \in dom \phi_x \), and \( \phi_x \) denotes the maximal solution to (7) starting from \( (x_0, q_0, r) \), implying a feasible solution to Problem 2 for \( r \), \( \phi_x(k, j) \) and \( \phi_{v_{\text{prev}}}(k, j) \).

C. Finite-time Reachability

Next, we introduce a result demonstrating that the system is assured to enter the neighborhood of reference \( r \) within a finite time frame, guaranteeing the activation of \( \gamma_2 \). The following assumption asserts that \( c_2 \) is large enough such that the sublevel set \( T_{1,0}(r) \) contains the equilibria set for \( P_1 \).

Assumption 5.6: For any \( r \in \mathbb{R}^m \), the algorithm parameter \( c_2 \) is such that there exists a positive constant \( \delta_0 \) such that \( A_1(r) + \delta_0 \mathbb{B} \subseteq T_{1,0}(r) \).

Then we show that when \( P_1 \) is active, the value of the Lyapunov function for \( P_2 \) decreases to \( c_2 \) within finite time.

Theorem 5.7: (Finite-time reachability to \( T_{1,0} \)) Suppose Assumptions 5.7, 5.9, and 5.5 are satisfied. Given a reference \( r \in \mathbb{R}^m \) and \( (x_0, q_0) \in \mathbb{R}^n \times \{1, 2\} \) satisfying Assumption 5.4, there exists a discrete time instance \( k^* \in \mathbb{N} \) such that \( \phi_x(k^*, j^*) \in T_{1,0}(r) \), where \( \phi_x \) denotes the maximal solution to (7) starting from \( (x_0, q_0, r) \), \( j^* \in \{0, 1\} \) such that \( (k^*, j^*) \in dom \phi_x \), and \( T_{1,0}(r) \) is defined in (8).

D. Forward Invariance

Next, we present the result showing that if the system starts within \( T_{1,0}(r) \), then \( \gamma_2 \) helps maintain the system within \( T_{1,0}(r) \) forever, preventing any future jumps.
The state trajectory of Actual reference
-0.05
0.05
0.2
0.5
0.6
lim
3.1, 3.3 and 5.6 are satisfied. Given a reference \( r \) and \( q \) jumps can occur during the control process, preventing the solution to (7) starting from \((x_0, q_0, r)\).

E. Finite Number of Jumps

Subsequently, we demonstrate that a maximum of two jumps can occur during the control process, preventing the shifting phenomenon caused by design.

Theorem 5.9: Suppose Assumptions 3.1, 3.3 and 5.6 are satisfied. Given a reference \( r \in \mathbb{R}^n \) and \((x_0, q_0) \in \mathbb{R}^n \times \{1, 2\} \) satisfying Assumption 5.4, then \( \phi(x, j) \in T_1(r) \) for all \((k, j) \in \text{dom } \phi\), where \( \phi \) denotes the maximal solution to (7) starting from \((x_0, q_0, r)\).

F. Convergence of Virtual Reference to Reference

The following theorem guarantees that the virtual reference \( v \) converges to \( r \). Given that \( z \) converges to \( v \), it ensures that \( z \) converges to \( r \).

Theorem 5.10: (Convergence of \( v \) to \( r \)) Suppose Assumptions 3.1, 3.3 and 5.6 are satisfied. Given a reference \( r \in \mathbb{R}^n \) and \((x_0, q_0) \in \mathbb{R}^n \times \{1, 2\} \) satisfying Assumption 5.4 then \( \lim_{k \to \infty} \kappa(x, j) = r \) where \( \phi \) denotes the maximal solution to (7) starting from \((x_0, q_0, r)\).

VI. SIMULATION RESULTS

(a) The state trajectory of \( x_1 \). (b) The state trajectory of \( x_2 \).

Fig. 2: The system state trajectory using switched RG.

(a) The state trajectory of \( x_1 \). (b) The state trajectory of \( x_2 \).

Fig. 3: The system state trajectory using a single controller.

In this simulation, the switched RG algorithm is used to track the piecewise-constant reference signal for the double-integrator system. The Lyapunov function used in the simulation is \( V(x, v) = \begin{pmatrix} x - v \end{pmatrix}^T \begin{pmatrix} 6.8283 & 2.2626 \\ 2.2626 & 2.1010 \end{pmatrix} \begin{pmatrix} x - v \end{pmatrix} \).

The algorithm parameters \( c_1 \) is set as 0.2 and \( c_2 \) is set as 0.1. The maximal output admissible sets \( S_1^c \) and \( S_2^c \) are computed using the MPT3 toolbox [15]. The constraint imposed on \( x_1 \) is \( x_1 \in [-0.6, 0.6] \). The constraint imposed on \( x_2 \) is \( x_2 \in [-0.1, 0.1] \). The system initiates at the point \((0.6, 0)\), posing a challenge given that the initial state nearly breaches the constraints. Compared to using a single RG, the switched RG achieves rapid and non-oscillatory convergence to the reference signal, as depicted in Figures 3a and 3b. Feasible results are obtained.

Theorem 5.8: (Forward invariance) Suppose Assumptions 3.1, 3.3 and 5.6 are satisfied. Given a reference \( r \in \mathbb{R}^n \) and \((x_0, q_0) \in \mathbb{R}^n \times \{1, 2\} \) satisfying Assumption 5.4 and \( x_0 \in T_0(r) \) where \( T_0(r) \) is defined in \( S \), then \( \phi(x, j) \in T_1(r) \) for all \((k, j) \in \text{dom } \phi\), where \( \phi \) denotes the maximal solution to (7) starting from \((x_0, q_0, r)\).

VII. CONCLUSION

In this paper, we proposed a switched RG approach for rapid, non-oscillatory convergence to a given reference signal while satisfying constraints, driven by a Lyapunov function. We demonstrate robust switching, recursive feasibility, and convergence of the virtual reference to the reference, alongside other key properties of the proposed switched RG.

REFERENCES