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Evolution of Polar Coding

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Abstract

Survey of polar coding towards our QC polar codes, and analyze for wireless communications channel.

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Evolution of Polar Coding

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***Abstract*—We first overview three decades of progress in channel coding research. Specifically, we focus on the research trends of polar codes in comparison to other competitive codes such as turbo codes and low-density parity-check (LDPC) codes. We then introduce our novel approach called quasi-cyclic (QC) polar codes, which can eliminate short cycles in the code graph. We demonstrate the effective design of QC polar codes with protograph extrinsic information transfer (P-EXIT) analysis to reduce the error floors of belief-propagation decoding.**

I. Introduction

Capacity-achieving forward error correction (FEC) techniques based on low-density parity-check (LDPC) codes have made great contributions in increasing data rates for modern wireless and optical communications systems. However, the pursuit of high FEC performance has led to a significant increase in power consumption and circuit size. Attaining a good trade-off between performance and computational complexity is of great importance. In addition, high-performance LDPC codes usually require large codeword lengths, whereas shorter FEC codes are preferred for latency-constrained systems, such as Internet-of-Things (IoT) applications.

Recently, polar codes [1] have drawn much attention as alternative capacity-achieving codes in place of LDPC codes for short block lengths, in particular for the fifth-generation (5G) networks and beyond. In this paper, we briefly overview technical milestones in the past decades, such as successive cancellation list (SCL) decoding [3], to analyze the evolution of polar codes. We then introduce a few techniques to improve the tradeoff between performance and complexity for high-throughput communications. In particular, we focus on recently proposed protograph codes called quasi-cyclic (QC) polar codes [14], that enable belief propagation (BP) decoding to perform well by eliminating short cycles in the code graphs. QC polar codes have various advantages over the regular counterpart; e.g., lower complexity, higher gain, lower latency, and higher parallelism. Specifically, it can build an attractive bridge between polar codes and LDPC codes since most techniques introduced for QC LDPC codes can be also applied to QC polar codes. To demonstrate its capability, we investigate the use of protograph extrinsic information transfer (P-EXIT) analysis [15], originally proposed for designing QC LDPC codes, to design QC polar codes. We also provide some insights for the future advancement of polar codes.

II. Evolution of Polar Codes

A. Research Trends

We first would like to discuss a trend of channel coding in the past decades. Fig. 1 shows the number of articles per year, counted based on the keyword hits in Google Scholar (excluding patents) for 1990–2020. The keywords we chose are "convolutional codes," "turbo codes," "LDPC codes," "polar codes," "repeat-accumulate codes," "rateless codes," "LDGM codes," and "staircase codes."

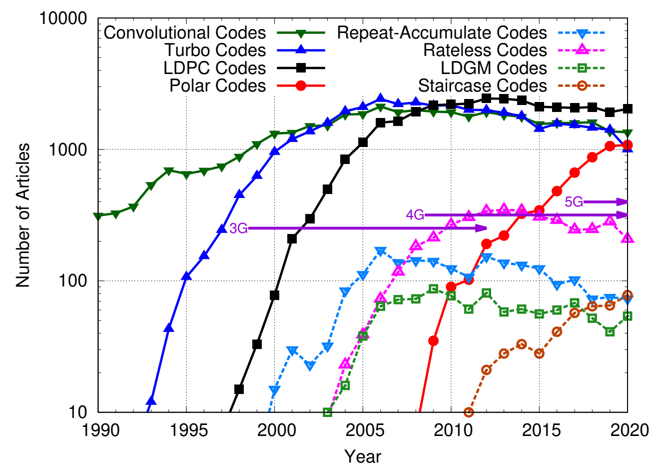

Fig. 1. Annual number of articles (Google Scholar keyword hit) discussing a family of channel coding over the last three decades.

In the first decade of 1990–2000, convolutional codes had been established as mainstream codes, with the number of papers growing to reach more than 1,000 articles annually. This article growth was partly promoted by the advent of turbo codes—practical capacity-achieving codes—in the early 90's. In the later 90's, LDPC codes were rediscovered as alternative capacity-achieving codes, and the article count growth of LDPC codes followed that of turbo codes with a time lag of approximately 5 years. During the next decade of 2000–2010, the quantity of literature on convolutional/turbo codes reached a peak in 2006, and has shown a decreasing trend since then. Subsequently, turbo codes relinquished the lead to LDPC codes in 2009. Around the same year, polar codes emerged as new capacity-achieving codes. The number of articles on polar codes has increased in the last decade; exceeding 1,000 per year since 2019 and overtaking turbo codes in 2020. Although LDPC codes are presently still mainstream codes, its volume of articles has been decreasing since their peak in 2012.

For the other four codes, the annual number of articles has been far below 1,000, and they have already passed a peak year, except for staircase codes. It should be noted that the trends have been greatly influenced by industrial standard activities; e.g., staircase codes have been recently adopted in

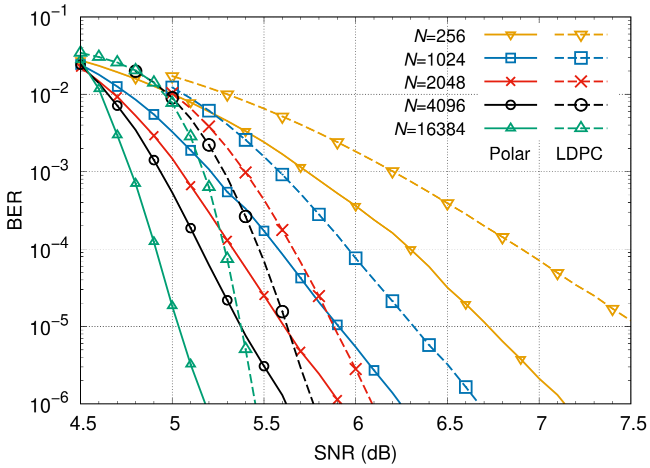

Fig. 2. BER comparison of LDPC codes with 4-iteration layered BP decoding and polar codes with 4-list SCL decoding for various block lengths in AWGN channels (code rates: $R = 0.8$) [37].

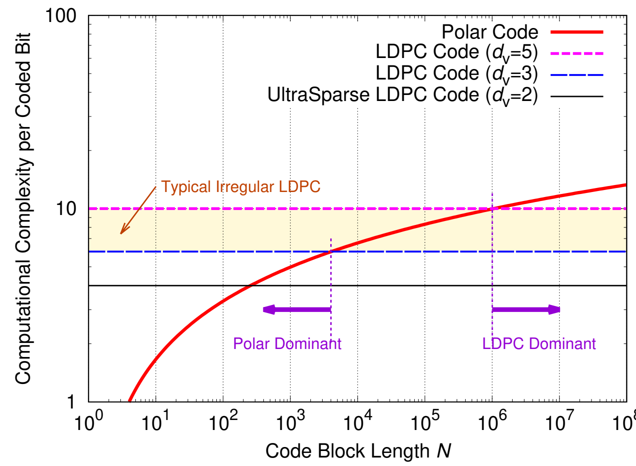

Fig. 3. Computational complexity per coded bit as a function of block length N for polar SCL decoding (per list) and LDPC BP decoding (per iteration).

400ZR standards, which might support their growth. Turbo codes have been widely used in the 3G/4G standards since the 2000–2010 decade, and LDPC/polar codes have been adopted in the 5G standards for the last decade. From the figure, it is implied that polar codes can be the most promising candidates to overtake the mainstream of channel coding, replacing LDPC codes in the future. Nonetheless, the growth from 2019 to 2020 is very marginal. This suggests that further practical applications and technical breakthroughs may be required to boost the future advancement.

Note that the keyword hit may not be a precise indicator to analyze the research trends since some articles just addressed the corresponding words without any technical investigations. Nevertheless, we hope that the above discussion provides some useful insights to understand the history.

B. Why Polar Codes?

LDPC codes offer excellent performance at large block lengths, while the performance of short codes is often poor. For this reason, polar codes are more viable candidates for short-packet transmission in latency-critical applications. As reported in [2]–[4], polar codes can outperform turbo codes and LDPC codes that are used in wireless standards when SCL decoding is adopted. Fig. 2 compares bit-error rate (BER) performance of state-of-the-art LDPC codes and polar codes for relatively short block lengths N. Here, we consider low-complexity cases with $I = 4$ iterations for LDPC layered BP decoding, and a list size of $L = 4$ for polar SCL decoding. LDPC codes perform worse than polar codes for all block lengths we considered. LDPC codes require more BP iterations and longer block lengths to compete well against polar codes.

In addition to the coding gain, polar codes are advantageous in the computational complexity. It was believed that the nonlinear complexity of polar SCL decoding, i.e., $\mathcal{O}[LN \log_2(N)/2]$, is a major drawback in comparison to the linear complexity of LDPC BP decoding, i.e., $\mathcal{O}[2Id_{\mathrm{v}}N]$ where d_{v} denotes the average variable-node degree. However, it turns out to be an advantage when we aim to reduce the block sizes in order to decrease decoding latency. This is illustrated in Fig. 3, where complexity per coded bit is plotted as a function of block length N for polar and LDPC decoding. Remarkably, polar decoding will be more efficient than typical LDPC decoding at short block lengths of $N < 10^4$. These results in Figs. 2 and 3 confirm that polar codes can be a strong FEC candidate for latency- and power-constrained systems.

C. Technical Milestones

Although there exist nearly 6,000 articles on polar codes as of today, we would like to highlight a few technical milestones in the evolution of polar codes as listed in Table I.

1) Polar Kernel: Arıkan proved that the multi-stage application of a simple kernel produces the so-called polarization phenomenon that yields capacity-approaching performance in arbitrary memoryless binary channels [1]. It was later found that a larger kernel can improve the error exponent [31]–[33]. Nonbinary [30] and nonlinear kernels [33] were also investigated and shown to achieve capacity. Exploiting multiple kernels at different polarization stages were then proposed [34], [35]. Further exploitation of irregular kernels was explored for complexity and latency reduction [36], [37].

2) Frozen Bit: Frozen bit locations were originally designed via the Bhattacharyya parameter [1]. To analyze the precise bit reliability, density evolution (DE) [19] was introduced, and later simplified by quantization [18], Gaussian approximation (GA) [20], and EXIT [10]. As a more simplified approach, beta expansion [21] gained much attention in the industrial community. Recently, data-driven methods such as genetic algorithms [22] and deep learning [23] were proposed.

3) Decoding Algorithm: The original SC decoding [1] was improved by the SCL decoder [3] as the most brilliant milestone. The combination of SCL decoding with an embedded cyclic redundancy check (CRC) to reject invalid paths yields significantly improved performance [38]. Various other

TABLE I
SAMPLE TECHNICAL MILESTONES IN THE EVOLUTION OF POLAR CODING

Kernel design	Large/nonlinear kernel [31]–[33], mixed/multi-kernel [34], [35], nonbinary [13], [30]
Frozen bit design	Bhattacharyya [1], DE [18], [19], GA [20], beta expansion [21], genetic alg. [22], deep learning [23], EXIT [9]
Decoding algorithm	SC [1], SCL [3], stack [24], simplified SC [25], neural net [26], [27], BPL [28], adaptive list [4], flip [29]
High coding gain	Interleaver [9], BICM-ID [11], concatenation [38]–[40], subcode [41], turbo product [43], protograph [14]
Low-complexity	Relaxed [36], irregular [37], simplified min-sum, quantization [5], [6], lookup table [7], analog decoding [51]
Rate adaptation	Rate compatible [44], universal [47], puncturing [45], shortening [46], adaptive modulation, nested, rateless, augment [48]
Application	5G [49], broadcast [53], relay [54], MAC [55], secrecy [50], molecular comm. [51], source coding [52], shaping [12]

decoding algorithms based on SC decoding were proposed in the literature, e.g., simplified SC [25], SC stack [24], neural SC [26], and SC flip decoding [29]. Although BP decoding performs poorly for polar codes, there are some improvement methods, e.g., BP list (BPL) decoding [28].

4) Application: Besides the 5G applications [49], polar codes have been used for broadcast [53], relay [54], wiretap [50], and multiple access channels (MAC) [55]. It was also shown that polar codes can be optimal for source coding [52].

D. Contributions

The authors have also contributed to some evolution of polar codes in the past years, e.g., as follows:

1) Polar-Coded Modulation: The conventional polar code construction assumes memoryless identical channel reliability, which is often not valid when we use high-order modulation formats. Consequently, careful interleaver design that properly maps the coded bits to modulation bit-planes is important to prevent destroying the channel polarization. We have optimized [9] an interleaver for bit-interleaved coded modulation (BICM) systems. It was extended to frequency-selective fading channels to exploit higher diversity gain [9]. In [12], a joint probabilistic shaping and polar coding was realized without an external distribution matcher, through the use of dynamic frozen bits. A nonbinary-input coded modulation was investigated with nonbinary polar codes, achieving about 1 dB gain over BICM. In addition, polar BICM with iterative demodulation (ID) was studied in [11].

2) Irregular Polar Codes: We have proposed irregular polar codes, whose polarization units are irregularly pruned to reduce the computational complexity and decoding latency [10], [37]. We also proposed to use EXIT analysis for joint frozen, interleaver, and pruning design. The proposed irregular polar codes showed slightly better performance than regular counterparts while reducing decoding complexity by at least 30% and decoding latency by 80%.

3) Turbo Polar Codes: One major drawback of polar SCL decoding lies in the difficulty of parallel implementation, leading to low throughput. To tackle this issue, we have proposed polar turbo product codes (TPC) [11], [43] constituting spatially-coupled parallel short-block polar codes. With parallel and pipeline SCL decoding, the proposed polar-TPC $(256, 239)^2$ achieves roughly 256-times faster decoding throughput. The polar-TPC offers a 0.5 dB gain over the conventional TPC based on BCH codes.

4) Hardware-Oriented Polar Decoding: In practical hardware, we generally need to consider finite-precision operations, which can be fixed-point arithmetic for typical energy-efficient interfaces. We have proposed an optimized non-uniform quantization [7] to improve the resilience against precision errors, exploiting the quantized DE [8] to design the lookup table (LUT). Also, we developed a soft-output decoding based on SCL for turbo decoding [43]. We further introduced a bio-chemical polar decoder for molecular communications [51].

5) Protograph Polar Codes: Most recently, we have proposed a new family called protograph-based QC polar codes [14]. The proposed QC polar codes offer a breakthrough to resolve the issue that BP decoding does not work well for polar codes due to short cycles. We can fully parallelize QC polar codes whose complexity is reduced to $\mathcal{O}[LN\log_2(N/Q)/2]$ for a lifting factor of Q. It was demonstrated that QC polar codes can outperform the standard polar codes while the computational complexity can be significantly reduced.

III. QC POLAR CODES

While we have shown a great potential of QC polar codes in [14], many fascinating topics have emerged to explore the full capability. In this paper, we address how to design QC polar codes by making use of P-EXIT [15].

A. Protograph Polar Codes

The concept of protograph codes, originally proposed for LDPC codes, was first introduced to polar codes in [14]. In a manner analogous to the lifting operation for protograph LDPC codes, we amend the generator matrix of polar codes. For example, the following generator matrix for 2-stage polar codes is modified with permutation matrices $\boldsymbol{P}_{i,j}$:

$$\boldsymbol{G}^{\otimes 2} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\text{Lifting}}{\Longrightarrow} \begin{bmatrix} \boldsymbol{P}_{1,1} & \boldsymbol{P}_{1,2} & \boldsymbol{P}_{1,3} & \boldsymbol{P}_{1,4} \\ \mathbf{0} & \boldsymbol{P}_{2,2} & \mathbf{0} & \boldsymbol{P}_{2,4} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{P}_{3,3} & \boldsymbol{P}_{3,4} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{P}_{4,4} \end{bmatrix},$$

where $\mathbf{0}$ is an all-zero matrix of size $Q \times Q$. QC polar codes use a weight-1 circulant permutation matrix: $\boldsymbol{P}_{i,j} = \boldsymbol{I}(s'_{i,j})$, where $\boldsymbol{I}(s)$ denotes the sth circulant permutation matrix obtained by cyclically right-shifting a $Q \times Q$ identity matrix by s positions, and $s'_{i,j}$ is a shift value to design. The permutation size Q is called the lifting size. In our lifting operation, we replicate Q-parallel polar encoders and permute exclusive-or incident bits among the parallel encoders at every proto-polarization unit as illustrated in Fig. 4.

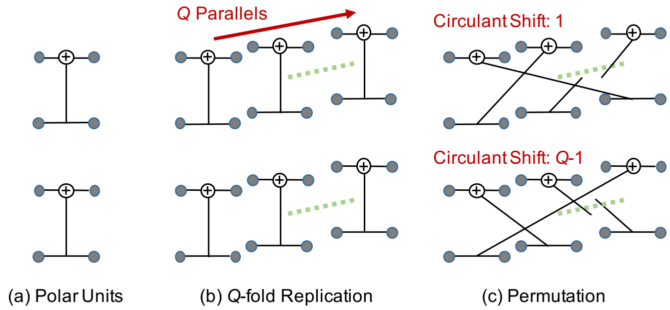

Fig. 4. Lifting operation for proto-polarization units: (a) regular polar units, (b) replication of Q-parallel encoders, (c) permutation for interleaving intermediate encoding bits at every proto-polarization unit [14].

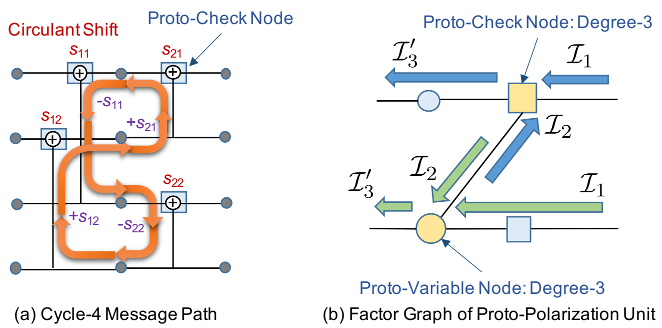

Fig. 5. (a) Cycle-4 message passing loop, which can be eliminated with proper shift values [14]. (b) P-EXIT evolution in proto-polarization unit.

QC polar codes have various advantages such as a highly parallel structure, shallower polarization stages, lower complexity, and lower latency. More importantly, QC polar codes can eliminate short cycles to increase the girth of the factor graph by optimizing shift values as verified in [14]. In contrast, the factor graphs of standard polar codes are inherently loopy, and unfortunately there exist a large number of short cycles as depicted in Fig. 5(a), leading to poor BP performance.

B. P-EXIT Design

In this paper, we extend the design method to jointly optimize frozen bit locations and circulant shift values by means of P-EXIT [15]. The conventional methods [1], [18]–[20], that assume uniform bit reliability, are not suitable for optimizing QC polar codes when high-order modulations and BP decoding are used. We propose to introduce P-EXIT analysis [15] for tracking the non-uniform mutual information at every proto-polarization unit. P-EXIT design can be used for both SCL and BP decoding by considering proper scheduling.

As shown in Fig. 5(b), each proto-polarization unit contains a proto-check node and a proto-variable node, both of which are of degree-3. Letting $\mathcal{I}_j$ denote incoming mutual information from the jth edge, the outgoing mutual information are updated in P-EXIT analysis [15] as follows:

$$\begin{aligned} \mathcal{I}'_3 &= 1 - J\Big(\sqrt{[J^{-1}(1-\mathcal{I}_1)]^2 + [J^{-1}(1-\mathcal{I}_2)]^2}\Big), \\ \mathcal{I}'_3 &= J\Big(\sqrt{[J^{-1}(\mathcal{I}_1)]^2 + [J^{-1}(\mathcal{I}_2)]^2}\Big), \end{aligned} \tag{1}$$

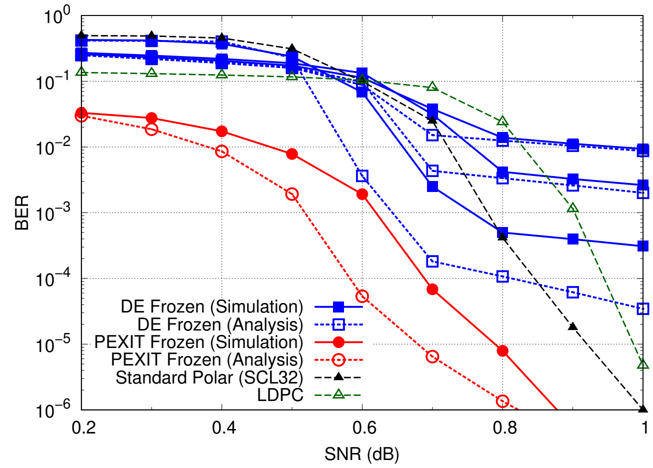

Fig. 6. BER performance of QC polar codes $(2^8, 2^7, 2^8)$ with $N = 2^{16}$ bits for 32-iteration BP decoding. Frozen bit location is designed by conventional DE [18] or P-EXIT [15]. Analytical results are based on P-EXIT analysis.

for the proto-check and proto-variable nodes, respectively. Here, $J(\cdot)$ is ten Brink's J-function [15] and $J^{-1}(\cdot)$ is its inverse function. We initialize the left-most incoming mutual information with 0 or 1 depending on frozen bit location, and the right-most incoming mutual information with the channel mutual information. We perform the P-EXIT evolution in (1) iteratively to trace the mutual information of each proto-polarization unit according to the decoder scheduling. Once the left-most outgoing mutual information $\mathcal{I}'_i$ is obtained, the error rate P_e is predicted as

$$P_\mathrm{e} = \frac{1}{|\mathbb{K}|}\sum_{i\in\mathbb{K}} \mathbb{Q}\Big(\frac{1}{2}J^{-1}(\mathcal{I}'_i)\Big), \tag{2}$$

where $\mathbb{Q}(\cdot)$ is defined as $\mathbb{Q}(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp(-t^2/2)\,\mathrm{d}t$. Here, $\mathbb{K}$ denotes the set of unfrozen bit locations.

C. Performance Analysis

Fig. 6 shows BER performance of half-rate QC polar codes with 8-stage polarization and $Q = 2^8$ lifting ($N = 2^{16}$ bits). We optimized frozen-bit location by the conventional DE [18], which assumes SC decoding, at a target signal-to-noise ratio (SNR) of 1, 3, and 12 dB. When sweeping the target SNR for DE design from 0 dB to 50 dB with 0.1 dB step, there were no better frozen-bit locations than that for 12 dB. Significant error floors were found, which is partly due DE [18] not assuming BP decoding. We also present the analytical BER performance for the corresponding frozen bits by using P-EXIT analysis. Remarkably, P-EXIT analysis agrees well the simulated performance especially for higher BER regimes, and the error floors are well predicted. With simulated annealing using P-EXIT, we can optimize the frozen-bit locations for BP decoding, whose performance is significantly improved over the conventional DE design by reducing the floors. It was also demonstrated that the optimized QC polar codes outperform state-of-the-art LDPC codes and standard polar codes.

IV. Conclusion and Discussion

We looked into technical trends of polar codes in the past decades, and discussed a recently proposed new family of polar coding, called QC polar codes. We investigated how to design QC polar codes by making use of P-EXIT analysis, which was originally introduced for QC LDPC codes. It was demonstrated that our design method based on P-EXIT can improve performance of QC polar codes in comparison to the conventional DE method. There remain many research directions stemming from QC polar codes, e.g., extensions to BPL decoding, systematic encoding, nonbinary codes, systematic circulant shift design, BP scheduling optimization, and multi-weight permutation. We envision that the polar-type protograph design for generalized LDGM (including QC polar codes) will stimulate the research community.

As we discussed, there were many milestones in the evolution of polar codes. Recognizing the fact that the research trends have been highly dependent on industrial activities such as the 5G standards, it is encouraged to apply polar codes to various applications for further advancement; e.g., bio sensing, molecular/THz communications, free-space optics, tactile network, mixed reality, data-center interconnects, deep space, secure/private network, green communications, LiFi, medical storage, quantum computing, etc.

References

[1] E. Arıkan, "Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels," *IEEE TIT*, vol. 55, no. 7, pp. 3051–3073, July 2009.

[2] G. Sarkis, et al., "Flexible and low-complexity encoding and decoding of systematic polar codes," *IEEE TCOM*, vol. 64, no. 7, July 2016.

[3] I. Tal and A. Vardy, "List decoding of polar codes," *IEEE TIT*, vol. 61, no. 5, pp. 2213–2226, May 2015.

[4] B. Li, H. Shen, and D. Tse, "An adaptive successive cancellation list decoder for polar codes with cyclic redundancy check," *IEEE COMML*, vol. 16, no. 12, pp. 2044–2047, Dec. 2012.

[5] S. H. Hassani and R. Urbanke, "Polar codes: Robustness of the successive cancellation decoder with respect to quantization," *ISIT*, 2012.

[6] Z. Shi, K. Chen, and K. Niu, "On optimized uniform quantization for SC decoder of polar codes," *VTC2014-Fall*, 2014.

[7] T. Koike-Akino, et al., "Hardware-efficient quantized polar decoding with optimized lookup table," *OECC*, July 2019.

[8] I. Tal and A. Vardy, "How to construct polar codes," *IEEE TIT*, vol. 59, no. 10, pp. 6562–82, Oct. 2013.

[9] T. Koike-Akino, et al., "Bit-interleaved polar-coded OFDM for low-latency M2M wireless communications," *ICC*, 2017.

[10] C. Cao, T. Koike-Akino, Y. Wang, and S. C. Draper, "Irregular polar coding for massive MIMO channels," *GLOBECOM*, 2017.

[11] T. Koike-Akino, et al., "EXIT convergence analysis of BICM-ID based on high-dimensional modulation and polar-TPC," *ECOC*, 2018.

[12] T. Matsumine, et al., "Polar-coded modulation for joint channel coding and probabilistic shaping," *OFC*, Mar. 2019.

[13] S. Cayci, T. Koike-Akino, and Y. Wang, "Nonbinary polar coding for multilevel modulation," *OFC*, Mar. 2019.

[14] T. Koike-Akino and Y. Wang, "Protograph-based design for QC polar codes," *ISIT*, July 2021.

[15] G. Liva and M. Chiani, "Protograph LDPC codes design based on EXIT analysis," *GLOBECOM*, Dec. 2007.

[16] D. M. Shin, S. C. Lim, and K. Yang, "Mapping selection and code construction for 2^m-ary polar-coded modulation," *IEEE COMML*, vol. 16, no. 6, pp. 905–908, June 2012.

[17] M. Seidl, A. Schenk, C. Stierstorfer, and J. B. Huber, "Polar-coded modulation," *IEEE TCOM*, vol. 61, no. 10, pp. 1302–6, Oct. 2013.

[18] I. Tal and A. Vardy, "How to construct polar codes," *IEEE TIT*, vol. 59, no. 10, pp. 6562–82, Oct. 2013.

[19] R. Mori and T. Tanaka, "Performance of polar codes with the construction using density evolution," *IEEE COMML*, vol. 13, no. 7, July 2009.

[20] P. Trifonov, "Efficient design and decoding of polar codes," *IEEE TCOM*, vol. 60, no. 11, pp. 3221–27, Nov. 2012.

[21] G. He, et al., "Beta-expansion: A theoretical framework for fast and recursive construction of polar codes," *GLOBECOM*, Dec. 2017.

[22] A. Elkelesh, M. Ebada, S. Cammerer, and S. ten Brink, "Decoder-tailored polar code design using the genetic algorithm," *IEEE TCOM*, vol. 67, no. 7, pp. 4521–34, Apr. 2019.

[23] M. Ebada, S. Cammerer, A. Elkelesh, and S. ten Brink, "Deep learning-based polar code design," *Allerton*, Sep. 2019.

[24] K. Niu and K. Chen, "Stack decoding of polar codes," *Electron. Lett.*, vol. 48, no. 12, pp. 695–7, June 2012.

[25] A. Alamdar-Yazdi and F. R. Kschischang, "A simplified successive-cancellation decoder for polar codes," *IEEE COMML*, vol. 15, no. 12, pp. 1378–80, Oct. 2011.

[26] N. Doan, S. A. Hashemi, and W. J. Gross, "Neural successive cancellation decoding of polar codes," *SPAWC*, June 2018.

[27] S. Cammerer, et al., "Scaling deep learning-based decoding of polar codes via partitioning," *GLOBECOM*, Dec. 2017.

[28] A. Elkelesh, et al., "Belief propagation list decoding of polar codes," *IEEE COMML*, vol. 22, no. 8, pp. 1536–9, June 2018.

[29] L. Chandesris, V. Savin, and D. Declercq, "Dynamic-SCFlip decoding of polar codes," *IEEE TCOM*, vol. 66, no. 6, pp. 2333–45, Jan. 2018.

[30] R. Mori and T. Tanaka, "Non-binary polar codes using Reed-Solomon codes and algebraic geometry codes," *ITW*, 2010.

[31] S. B. Korada, et al., "Polar codes: Characterization of exponent, bounds, and constructions," *IEEE TIT*, vol. 56, no. 12, pp. 6253–64, Nov. 2010.

[32] N. Presman, et al., "Binary polarization kernels from code decompositions," *IEEE TIT*, vol. 61, no. 5, pp. 2227–39, May 2015.

[33] H. P. Lin, S. Lin, K. A. Abdel-Ghaffar, "Linear and nonlinear binary kernels of polar codes of small dimensions with maximum exponents," *IEEE TIT*, vol. 61, no. 10, pp. 5253–70, Aug. 2015.

[34] N. Presman, et al., "Polar codes with mixed kernels," *ISIT*, July 2011.

[35] F. Gabry, et al., "Multi-kernel construction of polar codes," *ICC*, 2017.

[36] M. El-Khamy, et al., "Relaxed polar codes," *IEEE TIT*, vol. 63, no. 4, pp. 1986–2000, Apr. 2017.

[37] T. Koike-Akino, et al., "Irregular polar coding for complexity-constrained lightwave systems," *JLT*, vol. 36, no. 11, June 2018.

[38] Q. Zhang, A. Liu, X. Pan, and K. Pan, "CRC code design for list decoding of polar codes," *IEEE COMML*, vol. 21, no. 6, June 2017.

[39] Y. Wang, et al., "An improved concatenation scheme of polar codes with Reed–Solomon codes," *IEEE COMML*, vol. 21, no. 3, Dec. 2016.

[40] B. Li, et al., "RM-polar codes," *arXiv preprint* arXiv:1407.5483, 2014.

[41] P. Trifonov and V. Miloslavskaya, "Polar subcodes," *IEEE JSAC*, vol. 34, no. 2, pp. 254–66, Nov. 2015.

[42] B. Bourassa, M. Tremblay, and D. Poulin, "Convolutional polar codes on channels with memory," *arXiv preprint*, arXiv:1805.09378, May 2018.

[43] T. Koike-Akino, C. Cao, and Y. Wang, "Turbo product codes with irregular polar coding for high-throughput parallel decoding in wireless OFDM transmission," *ICC*, May 2018.

[44] K. Niu, K. Chen, and J. R. Lin, "Beyond turbo codes: Rate-compatible punctured polar codes," *ICC*, June 2013.

[45] R. Wang and R. Liu, "A novel puncturing scheme for polar codes," *IEEE COMML*, vol. 18, no. 12, pp. 2081–4, Oct. 2014.

[46] V. Bioglio, F. Gabry, and I. Land, "Low-complexity puncturing and shortening of polar codes," *WCNCW*, Mar. 2017.

[47] S. H. Hassani and R. Urbanke, "Universal polar codes," *ISIT*, June 2014.

[48] A. Elkelesh, M. Ebada, S. Cammerer, and S. ten Brink, "Flexible length polar codes through graph based augmentation," *SCC*, Feb. 2017.

[49] V. Bioglio, C. Condo, and I. Land, "Design of polar codes in 5G new radio," *IEEE Comm. Surveys & Tutorials*, Jan. 2020.

[50] H. Mahdavifar and A. Vardy, "Achieving the secrecy capacity of wiretap channels using polar codes," *IEEE TIT*, vol. 57, no. 10, Oct. 2011.

[51] T. Matsumine, et al., "Polar coding with chemical reaction networks for molecular communications," *GLOBECOM*, Dec. 2020.

[52] S. B. Korada and R. L. Urbanke, "Polar codes are optimal for lossy source coding," *IEEE TIT*, vol. 56, no. 4, pp. 1751–68, Mar. 2010.

[53] N. Goela, E. Abbe, and M. Gastpar, "Polar codes for broadcast channels," *IEEE TIT*, vol. 61, no. 2, pp. 758–82, Dec. 2014.

[54] M. Andersson, et al., "Nested polar codes for wiretap and relay channels," *IEEE COMML*, vol. 14, no. 8, pp. 752–4, Aug. 2010.

[55] E. Abbe and E. Telatar, "Polar codes for the m-user multiple access channel," *IEEE TIT*, vol. 58, no. 8, pp. 5437–48, May 2012.