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Mixed-Integer Linear Regression Kalman Filters for GNSS Positioning

Marcus Greiff[1], Karl Berntorp[1], Stefano Di Cairano [1], and Kyeong Jin Kim[1]

***Abstract*— In this paper, recursive filters are formulated for the mixed-integer GNSS receiver estimation problem, where the integer variables come from the ambiguities in the carrier-phase measurements. Insights from the linear setting illustrate pitfalls in designing optimal recursive filters, motivating a relaxation of the original optimization problem and a departure from conventional methods. A set of filters are developed for sequential nonlinear mixed-integer estimation based on statistical linearization, entertaining two estimate densities and taking the time-evolution of the ambiguities into account by adapting the process noise covariance based on a statistical model. Numerical examples illustrate the efficacy of the proposed algorithms.**

I. INTRODUCTION

The GNSS positioning problem concerns the estimation of a receiver's states from a set of code and carrier-phase measurements received from multiple satellites [1]. The involved measurement equations are time-varying, nonlinear, and include biases. In the carrier-phase measurements, there is an integer bias known as the *ambiguity*, unique to each carrier-phase measurement from each satellite [2]. These biases typically remain constant until sporadically and independently of each other jumping to new integer values, commonly referred to as "cycle slip". The works in [1], [3], [4] have highlighted performance gains that can be made if the ambiguity estimates are constrained to integers. These methods involve a first-order Taylor expansion of the measurement equation, with subsequent de-correlation and integer search methods to resolve the mixed-integer least-squares (MILS) problem. It has been adopted to an extended Kalman filter (EKF) setting (c.f., [2], [5]–[7]), used in GNSS receiver positioning (e.g., in the RTK library [8]). At each time step, a real-valued estimate is computed using a relaxed model, from whose distribution an integer ambiguity hypothesis is computed, before conditioning on this hypothesis. The problem has been well studied in the linear MILS-setting but there remains room for improvement in sequential filtering.

First, the design of sequential maximum-likelihood estimators for the mixed-integer GNSS positioning problem includes solving a mixed-integer problem of increasing size, and relaxations need to be made for the estimator to be implementable. In [5], [6], this is done by modeling a set of relaxed real-valued ambiguity states, and from that ambiguity hypothesis, the relaxed estimate distribution is corrected through a virtual measurement update [5]. This is a departure from the optimal solution in the linear setting in two important respects; (i), unlike the MILS-solution, the current integer ambiguity hypotheses are computed from a distribution conditioned on prior integer ambiguity hypotheses; and (ii), the random walk in the relaxed ambiguity dynamics of the filter prediction model does not resemble the integer jump process driving the cycle-slip behavior.

Second, depending on the nonlinearity of the flow and measurement equations in the estimation model, the explicit linearization in the EKF may be less accurate than the nonanalytic statistical linearizations of the linear-regression KFs (LRKFs) [9]. Certain LRKFs, such as the unscented KF (UKF), have been considered for GNSS positioning [10], but primarily for GNSS integration; that is, leveraging a GNSS module outputting position information and not using the measurements directly. One reason for this is that the higher-order LRKFs scale poorly with the number of states in the estimation model, a number that tends to be large in the context of the GNSS positioning problem. However, [11] showed that the LRKFs can be employed for problems with linear substructure, making them suitable for the moment approximations in sequential GNSS estimators.

We propose sequential maximum-likelihood estimators using statistical linearization for the GNSS positioning problem. First, we discuss the conventional MILS method in a recursive setting. We show that if the problem is to be solved exactly, its implementation becomes computationally intractable, with numerical complexity that grows significantly in time. Second, we propose a relaxation of the original problem by constraining past integer estimates, resulting in a recursive MILS algorithm, maximizing a likelihood while allowing the ambiguities to vary in time. Third, we extend the algorithm to a nonlinear setting, employing either explicit or statistical linearizations to evaluate the resulting moment integrals. Contrary to [2], [5]–[7], the presented methods require the storage of two different densities over the receiver states and will, in the static linear setting, result in a relaxed maximization of an associated log-likelihood cost.

Notation: Vectors are written as $\boldsymbol{x}$ with the i^{th} element denoted by x_i. Similarly, matrices are typeset as $\boldsymbol{A}$. The real numbers are denoted by $\mathbb{R}$ and the integers by $\mathbb{Z}$. The vector $\boldsymbol{1}_N \in \mathbb{R}^N$ is a column vector of ones, the matrix $\boldsymbol{0}_{N\times M} \in \mathbb{R}^{N\times M}$ is a zero matrix, $\boldsymbol{I}_N \in \mathbb{R}^{N\times N}$ is the identity matrix, and $\|\boldsymbol{x}\|_{\boldsymbol{A}}^2 = \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}$. In addition, $\mathcal{N}(\boldsymbol{x}|\boldsymbol{m}^{\boldsymbol{x}}, \boldsymbol{\Sigma}^{\boldsymbol{x}\boldsymbol{x}})$ denotes a Gaussian probability density function over $\boldsymbol{x}$ with mean $\boldsymbol{m}^{\boldsymbol{x}}$ and covariance $\boldsymbol{\Sigma}^{\boldsymbol{x}\boldsymbol{x}}$, following the notation in [12]. Random variables (RVs) associated with a density are denoted by $\underline{\boldsymbol{x}} \sim p_{\underline{\boldsymbol{x}}}(\boldsymbol{x})$, where the subscript of p is often omitted for brevity, and $\mathbb{E}[\underline{\boldsymbol{x}}] = \int \boldsymbol{x} p_{\underline{\boldsymbol{x}}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$. Let $(\cdot)_k$ be a variable at a time-step k, $(\cdot)_{a|b}$ a variable at a time a conditioned on information up to and including b. Symmetric matrix entries whose values are already evident, are indicated by $\star$.

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II. PRELIMINARIES

In the GNSS literature, the state of a receiver, partitioned into real-valued states $\boldsymbol{x}_k \in \mathbb{R}^m$ and integer ambiguities $\boldsymbol{n}_k \in \mathbb{Z}^n$, is commonly inferred from a linear measurement model with Gaussian noise $\underline{\boldsymbol{e}}_k \sim \mathcal{N}(\boldsymbol{e}_k|\boldsymbol{0}, \boldsymbol{R}_k)$, denoted by

$$\underline{\boldsymbol{y}}_k = \boldsymbol{H}_k\boldsymbol{x}_k + \boldsymbol{G}_k\boldsymbol{n}_k + \underline{\boldsymbol{e}}_k \in \mathbb{R}^p. \tag{1}$$

Commonly, inference is done by maximizing the measurement log-likelihood over the parameters $\boldsymbol{x}_k, \boldsymbol{n}_k$. That is, if

$$\mathcal{N}(\boldsymbol{y}_k|\boldsymbol{H}_k\boldsymbol{x}_k + \boldsymbol{G}_k\boldsymbol{n}_k, \boldsymbol{R}_k), \tag{2}$$

we seek to solve the weighted MILS problem on the form

$$\underset{\boldsymbol{x}_k\in\mathbb{R}^m, \boldsymbol{n}_k\in\mathbb{Z}^n}{\arg\min} \|\boldsymbol{y}_k - \boldsymbol{H}_k\boldsymbol{x}_k - \boldsymbol{G}_k\boldsymbol{n}_k\|^2_{\boldsymbol{R}_k^{-1}}. \tag{3}$$

As $\boldsymbol{R}_k = \boldsymbol{R}_k^\top \succ \boldsymbol{0}$, take $\boldsymbol{R}_k = \boldsymbol{L}\boldsymbol{L}^\top$ and define the linear maps $\boldsymbol{A}_k = \boldsymbol{L}_k^{-1}\boldsymbol{H}_k, \boldsymbol{B}_k = \boldsymbol{L}_k^{-1}\boldsymbol{G}_k$ and $\bar{\boldsymbol{y}}_k = \boldsymbol{L}_k^{-1}\boldsymbol{y}_k$. Then (3) can be equivalently written as an MILS problem

$$\underset{\boldsymbol{x}_k\in\mathbb{R}^m, \boldsymbol{n}_k\in\mathbb{Z}^n}{\arg\min} \|\bar{\boldsymbol{y}}_k - \boldsymbol{A}_k\boldsymbol{x}_k - \boldsymbol{B}_k\boldsymbol{n}_k\|^2_{\boldsymbol{I}}. \tag{4}$$

If we momentarily relax the above problem by lifting the integer constraint, with $\hat{\boldsymbol{x}}^f_k \in \mathbb{R}^m$, and $\hat{\boldsymbol{n}}^f_k \in \mathbb{R}^n$ denoting the relaxed estimates, these are given by the normal equations

$$(\boldsymbol{A}_k^\top\boldsymbol{A}_k)\hat{\boldsymbol{x}}^f_k + (\boldsymbol{A}_k^\top\boldsymbol{B}_k)\hat{\boldsymbol{n}}^f_k = \boldsymbol{A}_k^\top\bar{\boldsymbol{y}}_k, \tag{5a}$$
$$(\boldsymbol{B}_k^\top\boldsymbol{A}_k)\hat{\boldsymbol{x}}^f_k + (\boldsymbol{B}_k^\top\boldsymbol{B}_k)\hat{\boldsymbol{n}}^f_k = \boldsymbol{B}_k^\top\bar{\boldsymbol{y}}_k. \tag{5b}$$

If (i) the term $\boldsymbol{A}_k^\top\boldsymbol{A}_k$ is invertible, and (ii) there exists a solution to the normal equations in (5), then by completion of squares, the ordinary MILS problem in (4) can be written

$$\underset{\boldsymbol{x}_k\in\mathbb{R}^m, \boldsymbol{n}_k\in\mathbb{Z}^n}{\arg\min} \|\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k\|^2_{\boldsymbol{\Xi}_k^{-1}} + \|\boldsymbol{n}_k - \hat{\boldsymbol{n}}^f_k\|^2_{\Psi_k^{-1}}, \tag{6}$$

as originally pointed out in [13], where

$$\boldsymbol{\Xi}_k = (\boldsymbol{A}_k^\top\boldsymbol{A}_k)^{-1}, \tag{7a}$$
$$\boldsymbol{\Psi}_k = (\boldsymbol{B}_k^\top(\boldsymbol{I} - \boldsymbol{A}_k\boldsymbol{\Xi}_k\boldsymbol{A}_k^\top)\boldsymbol{B}_k)^{-1}, \tag{7b}$$
$$\hat{\boldsymbol{n}}^f_k = \boldsymbol{\Psi}_k\boldsymbol{B}_k^\top(\boldsymbol{I} - \boldsymbol{A}_k\boldsymbol{\Xi}_k\boldsymbol{A}_k^\top)\bar{\boldsymbol{y}}_k, \tag{7c}$$
$$\hat{\boldsymbol{x}}_k = \boldsymbol{\Xi}_k\boldsymbol{A}_k^\top(\bar{\boldsymbol{y}}_k - \boldsymbol{B}_k\boldsymbol{n}_k). \tag{7d}$$

The existence of (7b) is implied by (i) and (ii), as seen by the Shur complement. Thus, given (i) and (ii), we can compute $\boldsymbol{n}^I_k$ directly from the second term in (6), and then evaluate the estimate $\hat{\boldsymbol{x}}_k$ that zeros the first term in (6). Instead of dealing with the original (3), we only require the solution to

$$\boldsymbol{n}^I_k = \underset{\boldsymbol{n}_k\in\mathbb{Z}^n}{\arg\min}\|\boldsymbol{n}_k - \hat{\boldsymbol{n}}^f_k\|^2_{\boldsymbol{\Psi}_k^{-1}}, \tag{8}$$

which is NP-hard and therefore commonly solved using heuristics, typically involving de-correlation methods, such as the LAMBDA method in [3] or the MR-reduction in [14], and various integer search methods, such as the shrinking search-space or bootstrap methods outlined in [1] and [14].

III. PITFALLS WITH RECURSIVE ESTIMATORS

When considering recursive filtering, the integer-fixation (de-correlation and integer search) is often introduced heuristically. Part of the reason is that (8) is NP-hard, and needs to be solved over all prior ambiguities, therefore growing in size with time. For a recursive LS (RLS) scheme for the case of a static receiver position, if we define

$$\bar{\boldsymbol{Y}}_k = \begin{bmatrix}\bar{\boldsymbol{Y}}_{k-1}\\ \bar{\boldsymbol{y}}_k\end{bmatrix}, \quad \mathcal{A}_k = \begin{bmatrix}\mathcal{A}_{k-1}\\ \boldsymbol{A}_k\end{bmatrix}, \quad \mathcal{B}_k = \begin{bmatrix}\mathcal{B}_{k-1} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{B}_k\end{bmatrix},$$
$$\boldsymbol{N}_k = \begin{bmatrix}\boldsymbol{N}_{k-1}\\ \boldsymbol{n}_k\end{bmatrix}, \quad \boldsymbol{N}^I_k = \begin{bmatrix}\boldsymbol{N}^I_{k-1}\\ \boldsymbol{n}^I_k\end{bmatrix}, \quad \hat{\boldsymbol{N}}^f_k = \begin{bmatrix}\hat{\boldsymbol{N}}^f_{k-1}\\ \hat{\boldsymbol{n}}^f_k\end{bmatrix},$$

using (6) and (7), it can be shown that the solution to

$$\{\hat{\boldsymbol{x}}_k, \boldsymbol{N}^I_k\} = \underset{\boldsymbol{x}_k\in\mathbb{R}^m, \boldsymbol{N}_k\in\mathbb{Z}^{kn}}{\arg\min} \|\bar{\boldsymbol{Y}}_k - \mathcal{A}_k\boldsymbol{x}_k - \mathcal{B}_k\boldsymbol{N}_k\|^2_{\boldsymbol{I}}, \tag{9}$$

at time step k can lead to a revision of previous integer ambiguity estimates. To compute the solution to (9) exactly, we herefore need to solve increasingly larger ILS problems.

With some relaxations, we can approximately maximize the measurement likelihood in (9) by solving for the state distribution and integer ambiguity hypothesis at time step k. This is referred to as constrained RMILS, and is akin computing the most likely estimate conditioned on a past ambiguity trajectory. However, the resulting estimates will only approximately minimize the true MILS problem.

A. A Bayesian intepretation

The resulting algorithm can be given a Bayesian interpretation which will be useful in deriving the recursive filters. To start, we assume the solution to the relaxed LS-problem

$$\underset{\boldsymbol{x}_{k-1}\in\mathbb{R}^m, \boldsymbol{N}_{k-1}\in\mathbb{R}^{n(k-1)}}{\arg\min} \|\bar{\boldsymbol{Y}}_{k-1} - \mathcal{A}_{k-1}\boldsymbol{x}_{k-1} - \mathcal{B}_{k-1}\boldsymbol{N}_{k-1}\|^2_{\boldsymbol{I}}$$

is known at time step $k-1$ given the information $\bar{\boldsymbol{Y}}_{k-1}$. For this discussion, we are only interested in the distribution of this relaxed state estimate $\boldsymbol{x}^f_{k-1}$,

$$p(\boldsymbol{x}^f_{k-1}|\bar{\boldsymbol{Y}}_{k-1}) = \mathcal{N}(\boldsymbol{x}^f_{k-1}|\boldsymbol{m}^{\boldsymbol{x}^f}_{k-1}, \boldsymbol{\Sigma}^{\boldsymbol{x}^f\boldsymbol{x}^f}_{k-1}). \tag{10}$$

Using (10) as a prior for the real-valued state estimate at k, and no prior for the unconstrained ambiguities at k, we solve

$$\underset{\boldsymbol{x}_k\in\mathbb{R}^m, \boldsymbol{n}_k\in\mathbb{R}^n}{\arg\min} \|\bar{\boldsymbol{y}}_k - \boldsymbol{A}_k\boldsymbol{x}_k - \boldsymbol{B}_k\boldsymbol{n}_k\|^2_{\boldsymbol{I}} + \|\boldsymbol{x}_k - \boldsymbol{m}^{\boldsymbol{x}^f}_{k-1}\|^2_{(\boldsymbol{\Sigma}^{\boldsymbol{x}^f\boldsymbol{x}^f}_{k-1})^{-1}}, \tag{11}$$

yielding, through the normal equations (5), a solution

$$p(\boldsymbol{x}^f_k, \boldsymbol{n}^f_k|\bar{\boldsymbol{Y}}_k) = \mathcal{N}\left(\begin{bmatrix}\boldsymbol{x}^f_k\\ \boldsymbol{n}^f_k\end{bmatrix}\middle|\begin{bmatrix}\boldsymbol{m}^{\boldsymbol{x}^f}_{k|k}\\ \boldsymbol{m}^{\boldsymbol{n}^f}_{k|k}\end{bmatrix}, \begin{bmatrix}\boldsymbol{\Sigma}^{\boldsymbol{x}^f\boldsymbol{x}^f}_{k|k} & \boldsymbol{\Sigma}^{\boldsymbol{x}^f\boldsymbol{n}^f}_{k|k}\\ \star & \boldsymbol{\Sigma}^{\boldsymbol{n}^f\boldsymbol{n}^f}_{k|k}\end{bmatrix}\right). \tag{12}$$

If we are only interested in solving for the ambiugities at time step k, we can solve

$$\boldsymbol{n}^I_k = \underset{\boldsymbol{n}_k\in\mathbb{Z}^n}{\arg\min}\|\boldsymbol{n}_k - \boldsymbol{m}^{\boldsymbol{n}^f}_{k|k}\|^2_{(\boldsymbol{\Sigma}^{\boldsymbol{n}^f\boldsymbol{n}^f}_{k|k})^{-1}}. \tag{13}$$

With this new integer hypothesis, we seek to evaluate the distribution of the state conditioned on all prior measurements and integer ambiguities, $p(\boldsymbol{x}_k|\bar{\boldsymbol{Y}}_k, \boldsymbol{N}^I_k)$. It is difficult to use the density in (12) for this purpose, as it is only conditioned on the measurements and not prior ambiguity hypotheses. Instead, we can recursively update a separate distribution. We introduce an *integer constrained measurement*, as

$$\bar{\boldsymbol{z}}_k = \bar{\boldsymbol{y}}_k - \boldsymbol{B}_k \boldsymbol{n}_k^I \in \mathbb{R}^p, \tag{14}$$

and let $\bar{\boldsymbol{Z}}_k = (\bar{\boldsymbol{z}}_1^\top, ..., \bar{\boldsymbol{z}}_k^\top)^\top$. This leads to

$$p(\boldsymbol{x}_k|\bar{\boldsymbol{Z}}_{k-1}) = \mathcal{N}(\boldsymbol{x}_k|\hat{\boldsymbol{x}}_{k-1}, \Xi_{k-1}), \tag{15}$$

where Ξ_k can be computed recursively as

$$\Xi_k = \Xi_{k-1} - \Xi_{k-1}\boldsymbol{A}_k^\top(\boldsymbol{I} + \boldsymbol{A}_k\Xi_{k-1}\boldsymbol{A}_k^\top)^{-1}\boldsymbol{A}_k\Xi_{k-1}. \tag{16}$$

Given the measurement equation with a fixed integer ambiguity, we can form a joint one-step prediction

$$p(\boldsymbol{x}_k, \bar{\boldsymbol{z}}_k|\bar{\boldsymbol{Z}}_{k-1}) = \tag{17}$$
$$\mathcal{N}\left(\begin{bmatrix}\boldsymbol{x}_k\\ \bar{\boldsymbol{z}}_k\end{bmatrix}\middle|\begin{bmatrix}\boldsymbol{m}^{\boldsymbol{x}}_{k|k-1}\\ \boldsymbol{m}^{\boldsymbol{z}}_{k|k-1}\end{bmatrix}, \begin{bmatrix}\Xi_{k-1} & \Xi_{k-1}\boldsymbol{A}_k^\top\\ \star & \boldsymbol{I} + \boldsymbol{A}_k\Xi_{k-1}\boldsymbol{A}_k^\top\end{bmatrix}\right)$$

which, if conditioned on the constrained measurement $\bar{\boldsymbol{z}}_k$, yields the posterior distribution

$$p(\boldsymbol{x}_k|\bar{\boldsymbol{Z}}_k) = \mathcal{N}(\boldsymbol{x}_k|\hat{\boldsymbol{x}}_k, \Xi_k). \tag{18}$$

Thus, the approximate recursive solution to (9), when fixing the ambiguity hypotheses, has a Bayesian interpretation and requires the storage and recursive updating of two related but separate densities, $p(\boldsymbol{x}_k^f, \boldsymbol{n}_k^f|\bar{\boldsymbol{Y}}_k)$ and $p(\boldsymbol{x}_k|\bar{\boldsymbol{Z}}_k)$.

Remark 1 *If we intend to approximately solve the problem defined by the cost in* (9)*, with the assumption that we do not revise old ambiguity hypotheses, then this equates to storing two separate state estimate distributions; one for the solution of the relaxed problem, which is conditioned only on the measurements,* $p(\boldsymbol{x}_k^f, \boldsymbol{n}_k^f|\bar{\boldsymbol{Y}}_k)$*, and one for the approximate constrained problem, conditioned on all measurements and prior integer ambiguities,* $p(\boldsymbol{x}_k|\bar{\boldsymbol{Z}}_k)$*.*

Next, we use the above estimator as inspiration when deriving associated recursive filtering algorithms in the nonlinear setting, with dynamics and process noise, which will differ from the methods by which the ambiguity states are fixed in conventional GNSS filters. The propagation of both $p(\boldsymbol{x}_k^f, \boldsymbol{n}_k^f|\bar{\boldsymbol{Y}}_k)$ and $p(\boldsymbol{x}_k|\bar{\boldsymbol{Z}}_k)$ is essential to recover the optimal solution to the relaxed problem in the linear setting.

IV. FILTER FORMULATIONS

Given the Bayesian interpretation of the relaxed RMILS estimator, we extend the algorithm to a nonlinear Gaussian filtering setting using two different moment approximations: the first-order Taylor expansions in the EKF, and the statistical linearizations in the LRKFs.

A. Moment Approximations

In nonlinear filtering, a common way of approximating the moments of the output $\underline{\boldsymbol{o}}$ of a generic function $\boldsymbol{g}$, given a Gaussian input $\underline{\boldsymbol{i}} \sim p_{\underline{\boldsymbol{i}}}(\boldsymbol{i}) = \mathcal{N}(\boldsymbol{i}|\boldsymbol{m}^{\boldsymbol{i}}, \boldsymbol{\Sigma}^{\boldsymbol{ii}})$, is to explicitly linearize the function $\boldsymbol{g}$ about $\boldsymbol{m}^{\boldsymbol{i}}$, with

$$\boldsymbol{g}(\boldsymbol{i}) \approx \boldsymbol{g}(\boldsymbol{m}^{\boldsymbol{i}}) + \boldsymbol{G}(\boldsymbol{i} - \boldsymbol{m}^{\boldsymbol{i}}), \quad \boldsymbol{G} = \frac{\partial \boldsymbol{g}(\boldsymbol{i})}{\partial \boldsymbol{i}}\bigg|_{\boldsymbol{i}=\boldsymbol{m}^{\boldsymbol{i}}}. \tag{19}$$

The joint distribution of the output $\underline{\boldsymbol{o}} = \boldsymbol{g}(\underline{\boldsymbol{i}})$ and input $\underline{\boldsymbol{i}}$ can be approximated in its first two moments by

$$p_{\underline{\boldsymbol{io}}}(\boldsymbol{i}, \boldsymbol{o}) \approx \mathcal{N}\left(\begin{bmatrix}\boldsymbol{i}\\ \boldsymbol{o}\end{bmatrix}\middle|\begin{bmatrix}\boldsymbol{m}^{\boldsymbol{i}}\\ \boldsymbol{m}^{\boldsymbol{o}}\end{bmatrix}, \begin{bmatrix}\boldsymbol{\Sigma}^{\boldsymbol{ii}} & \boldsymbol{\Sigma}^{\boldsymbol{io}}\\ \star & \boldsymbol{\Sigma}^{\boldsymbol{oo}}\end{bmatrix}\right), \tag{20}$$

with the unknown moments in (20) given by

$$\boldsymbol{m}^{\boldsymbol{o}} \approx \boldsymbol{g}(\boldsymbol{m}^{\boldsymbol{i}}), \quad \boldsymbol{\Sigma}^{\boldsymbol{io}} \approx \boldsymbol{\Sigma}^{\boldsymbol{ii}}\boldsymbol{G}^\top, \quad \boldsymbol{\Sigma}^{\boldsymbol{oo}} \approx \boldsymbol{G}\boldsymbol{\Sigma}^{\boldsymbol{ii}}\boldsymbol{G}^\top. \tag{21}$$

Remark 2 *In the context of a nonlinear GNSS measurement equation,* $\underline{\boldsymbol{y}} = \boldsymbol{h}(\underline{\boldsymbol{x}}) + \boldsymbol{G}\underline{\boldsymbol{n}} + \underline{\boldsymbol{e}}$*, with Gaussian noise* $\underline{\boldsymbol{e}} \sim \mathcal{N}(\boldsymbol{e}|\boldsymbol{0}, \boldsymbol{R})$*, we could take the input to be all of the system states* $\boldsymbol{i} = (\boldsymbol{x}^\top, \boldsymbol{n}^\top)^\top$*, and let the output be the measurement vector,* $\underline{\boldsymbol{y}} = \underline{\boldsymbol{o}} = \boldsymbol{g}(\underline{\boldsymbol{i}}) + \underline{\boldsymbol{e}}$*. Assuming a Gaussian prior over the inputs, that the inputs and the noise are not correlated, the moments of the joint* $p_{\underline{\boldsymbol{x}},\underline{\boldsymbol{n}},\underline{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{y})$ *by* (21) *are then*

$$\boldsymbol{m}^{\boldsymbol{y}} = \boldsymbol{m}^{\boldsymbol{o}}, \quad \boldsymbol{\Sigma}^{\boldsymbol{iy}} = \boldsymbol{\Sigma}^{\boldsymbol{io}}, \quad \boldsymbol{\Sigma}^{\boldsymbol{yy}} = \boldsymbol{\Sigma}^{\boldsymbol{oo}} + \boldsymbol{R}. \tag{22}$$

An alternative to the explicit linearization in Remark 2 is the statistical linearization, where the moment integrals

$$\begin{aligned}
\boldsymbol{m}^{\boldsymbol{o}} &= \int_{\mathbb{R}^I} \boldsymbol{g}(\boldsymbol{i})\mathcal{N}(\boldsymbol{i}|\boldsymbol{m}^{\boldsymbol{i}}, \boldsymbol{\Sigma}^{\boldsymbol{ii}})d\boldsymbol{i}, \\
\boldsymbol{\Sigma}^{\boldsymbol{io}} &= \int_{\mathbb{R}^I} (\boldsymbol{i} - \boldsymbol{m}^{\boldsymbol{i}})(\boldsymbol{g}(\boldsymbol{i}) - \boldsymbol{m}^{\boldsymbol{o}})^\top\mathcal{N}(\boldsymbol{i}|\boldsymbol{m}^{\boldsymbol{i}}, \boldsymbol{\Sigma}^{\boldsymbol{ii}})d\boldsymbol{i}, \\
\boldsymbol{\Sigma}^{\boldsymbol{oo}} &= \int_{\mathbb{R}^I} (\boldsymbol{g}(\boldsymbol{i}) - \boldsymbol{m}^{\boldsymbol{o}})(\boldsymbol{g}(\boldsymbol{i}) - \boldsymbol{m}^{\boldsymbol{o}})^\top\mathcal{N}(\boldsymbol{i}|\boldsymbol{m}^{\boldsymbol{i}}, \boldsymbol{\Sigma}^{\boldsymbol{ii}})d\boldsymbol{i},
\end{aligned} \tag{23}$$

are approximated by evaluating $\boldsymbol{g}$ in a set of weighted integration points, $\mathcal{P} = \{(w^{(i)}, \boldsymbol{\xi}^{(i)})\}_{i=1}^{K(I)}$, where $I = \dim(\boldsymbol{i})$. This point set uniquely defines the various LRKFs, encompassing the unscented transform in [15], the spherical cubature rules in [16], and the Gauss-Hermite integration rules in [12]. The resulting finite-sum approximations of (23) are accurate to higher orders than the first-order approximation in (21), but the number of required integration points $K(I)$ often scales super-linearly or even exponentially in the number of input dimensions, I (c.f. [12]). However, [11] noted that linear substructure can be exploited to drastically decrease the number of integration points. If the measurement equation is be linear in a majority of the states, we evaluate the integrals in (23), using Proposition 1 in [11].

Remark 3 *Consider a nonlinear GNSS measurement equation,* $\underline{\boldsymbol{y}} = \boldsymbol{h}(\underline{\boldsymbol{x}}) + \boldsymbol{G}\underline{\boldsymbol{n}} + \underline{\boldsymbol{e}}$*, with Gaussian noise* $\underline{\boldsymbol{e}} \sim \mathcal{N}(\boldsymbol{e}|\boldsymbol{0}, \boldsymbol{R})$*. Decompose the real-valued states* $\boldsymbol{x}$ *into a nonlinear* $\boldsymbol{x}^n$ *and a linear part,* $\boldsymbol{x}^l$*, such that* $\boldsymbol{x} = ({\boldsymbol{x}^n}^\top, {\boldsymbol{x}^l}^\top)^\top$ *with* $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{h}^n(\boldsymbol{x}^n) + \boldsymbol{H}^l\boldsymbol{x}^l$*. Let* $\boldsymbol{i} = (\boldsymbol{x}^\top, \boldsymbol{n}^\top)^\top$ *and define*

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{G}\boldsymbol{n} + \boldsymbol{e} = \boldsymbol{T}\boldsymbol{g}(\boldsymbol{i}) + \boldsymbol{e}, \tag{24}$$

where

$$\boldsymbol{T} = \begin{bmatrix}\boldsymbol{I} & \boldsymbol{I}\end{bmatrix}, \qquad \boldsymbol{o} = \boldsymbol{g}(\boldsymbol{i}) = \begin{bmatrix}\boldsymbol{h}^n(\boldsymbol{x}^n)\\ \boldsymbol{H}^l\boldsymbol{x}^l + \boldsymbol{G}\boldsymbol{n}\end{bmatrix}. \tag{25}$$

Applying Proposition 1 in [11], the moments in (23) *can be computed given any LRKF, but now in* $K(\dim(\boldsymbol{x}^n))$ *function evaluations instead of* $K(\dim(\boldsymbol{i}))$ *evaluations. The approximate moments of* $p_{\underline{\boldsymbol{x}},\underline{\boldsymbol{n}},\underline{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{y})$ *are then given by*

$$\boldsymbol{m}^{\boldsymbol{y}} = \boldsymbol{T}\boldsymbol{m}^{\boldsymbol{o}}, \quad \boldsymbol{\Sigma}^{\boldsymbol{xy}} = \boldsymbol{\Sigma}^{\boldsymbol{xo}}\boldsymbol{T}^\top, \quad \boldsymbol{\Sigma}^{\boldsymbol{yy}} = \boldsymbol{T}\boldsymbol{\Sigma}^{\boldsymbol{oo}}\boldsymbol{T}^\top + \boldsymbol{R}. \tag{26}$$

B. Cycle slips and ambiguity dynamics

The integer ambiguities in the carrier-phase measurements need consideration when developing the filters. They likely remain constant over long periods of time before single ambiguities suddenly "jump" to a new integer values during a cycle slip event. This behavior can be modeled as a dynamical system driven by a special discrete random walk (here referred to as integer noise), an approach that differs from the modeling in [5], [6], where the ambiguities are assumed constant in time in the prediction model. To capture the ambiguity dynamics, consider a discrete stochastic process that with probability $b \in [0, 1]$ "jumps", attaining a random value drawn from a uniform distribution over the integers on the interval $I = [-a, a] \subset \mathbb{Z}$, and with probability $1 - b$ is zero. Let $\underline{s} \in I \subset \mathbb{Z}$ be an RV with the associated density

$$\mathcal{J}(s|a, b) = p_{\underline{s}}(s) = \begin{cases} (1-b)\delta(s-s_i) & \text{for } s_i = 0 \\ \frac{b}{2a}\delta(s-s_i) & \text{for } s_i \in I\backslash\{0\} \end{cases}. \quad (27)$$

where δ denotes Dirac's delta function. In the multivariate setting, the jumps are independent in each dimension, and on rectangular intervals about the origin defined by the corresponding elements of a vector $\boldsymbol{a} \in \mathbb{R}^n$ with jump probabilities defined by the elements of a vector $\boldsymbol{b} \in [0, 1]^n$. Here, the noise is a RV $\underline{\boldsymbol{s}} \in [-a_1, a_1] \times \cdots \times [-a_n, a_n] \subset \mathbb{Z}^n$, with the associated probability density function (PDF),

$$\underline{\boldsymbol{s}} \sim \mathcal{J}(\boldsymbol{s}|\boldsymbol{a}, \boldsymbol{b}) = p_{\underline{\boldsymbol{s}}}(\boldsymbol{s}) = \prod_{j=1}^{n} \mathcal{J}(s_j|a_j, b_j). \quad (28)$$

Using (28), we model the integer time-evolution

$$\underline{\boldsymbol{n}}_{k+1} = \underline{\boldsymbol{n}}_k + \underline{\boldsymbol{s}}_k, \qquad \underline{\boldsymbol{s}}_k \sim \mathcal{J}(\boldsymbol{s}_k|a\boldsymbol{1}, bh\boldsymbol{1}). \quad (29)$$

where $h > 0$, which implies a probability of an ambiguity jumping as b per ambiguity per time unit. The integer noise PDF in (27) and a realization of the jump process in (29) is used to generate the data in Sec. V.

C. Cycle-Slip Detection

Given ambiguity biases that evolve by (29), it is beneficial to capture the jump behavior in the ambiguity priors, as opposed to (11) which incorporates no such prior. One approach is to detect when a cycle slip occurs and adapt the priors for the relaxed estimation problem accordingly. The detection of the cycle slips can be done by considering some true state $\boldsymbol{x}_k$ and integer ambiguity $\boldsymbol{n}_k$, and an estimate distribution $p(\boldsymbol{x}_k^f, \boldsymbol{y}_k^f)$ with zero mean noise $\underline{\boldsymbol{e}}_k$, where

$$\underline{\boldsymbol{y}}_k = \boldsymbol{h}(\boldsymbol{x}_k) + \boldsymbol{G}_k\boldsymbol{n}_k + \underline{\boldsymbol{e}}_k \quad (30a)$$
$$\underline{\boldsymbol{y}}_k^f = \boldsymbol{h}(\underline{\boldsymbol{x}}_k^f) + \boldsymbol{G}_k\underline{\boldsymbol{n}}_k^f + \underline{\boldsymbol{e}}_k \quad (30b)$$

By defining $\delta\boldsymbol{y}_k = \mathbb{E}[\underline{\boldsymbol{y}}_k - \underline{\boldsymbol{y}}_k^f]$, $\delta\boldsymbol{n}_k = \mathbb{E}[\boldsymbol{n}_k - \underline{\boldsymbol{n}}_k^f]$, and assuming that all of the variation in the predicted measurement mean is due to the cycle slips, we obtain

$$\delta\boldsymbol{y}_k = \boldsymbol{G}_k\delta\boldsymbol{n}_k \Leftrightarrow \delta\boldsymbol{n}_k = (\boldsymbol{G}_k^\top\boldsymbol{G}_k)^{-1}\boldsymbol{G}_k\delta\boldsymbol{y}_k. \quad (31)$$

Note that for GNSS measurement equations, the matrix $\boldsymbol{G}$ generally has full column rank and the left-inverse in (31) always exists. If any dimension of $\delta\boldsymbol{n}_k$ is sufficiently far away from the origin, determined by a threshold d, we ascribe this to the presence of a cycle slip. In the event of such cycle slips, we are likely to see significant changes in the corresponding dimension of $\boldsymbol{n}_k$ between consecutive time steps, and the uncertainty in the prediction should therefore increase to reflect the parameter a of the integer jump noise. We capture this by defining an incidence vector $\boldsymbol{c}_k$, with

$$c_{i,k} = \begin{cases} 1 & \text{if } |\delta n_{i,k}| > d \\ 0 & \text{otherwise} \end{cases}, \quad (32)$$

and consider a prediction model in the ambiguity biases of the relaxed filter from time step $k-1$ to k as

$$\boldsymbol{n}_k = \boldsymbol{n}_{k-1} + \underline{\boldsymbol{v}}_{k-1}, \quad (33a)$$
$$\underline{\boldsymbol{v}}_{k-1} \sim \mathcal{N}(\boldsymbol{v}_{k-1}|\boldsymbol{0}, \boldsymbol{V}_{k-1}), \quad (33b)$$
$$\boldsymbol{V}_{k-1} = \mathrm{diag}(\boldsymbol{c}_k\sigma_{jump}^2 + (\boldsymbol{1} - \boldsymbol{c}_k)\sigma_{stay}^2), \quad (33c)$$

where the variances σ_{jump}^2 reflect the support of the integer jump process a, and σ_{stay}^2 is a regularizing term, modeling uncertainty in the relaxed estimate ambiguity dynamics when no cycle slip is detected. As a rule of thumb, we suggest letting $\sigma_{jump} \approx a$ and $0.01 < \sigma_{stay} < 0.4$.

D. Filters with adaptive ambiguity priors

Given the Bayesian interpretation of the RMILS algorithm in Sec. III, we now extend the algorithm to the nonlinear filtering setting using the moment approximations in either (21) or (23) to evaluate the joint distributions in (12) and (17). Simultaneously, we incorporate the adaptive ambiguity priors for the relaxed estimation problem, which reflect the behavior of the integer jump process in (29). Consider the model

$$\underline{\boldsymbol{x}}_{k+1} = \boldsymbol{f}_k(\underline{\boldsymbol{x}}_k) + \underline{\boldsymbol{w}}_k, \qquad \underline{\boldsymbol{w}}_k \sim \mathcal{N}(\boldsymbol{w}_k|\boldsymbol{0}, \boldsymbol{W}_k), \quad (34a)$$
$$\underline{\boldsymbol{n}}_{k+1} = \underline{\boldsymbol{n}}_k + \underline{\boldsymbol{v}}_k, \qquad \underline{\boldsymbol{v}}_k \sim \mathcal{N}(\boldsymbol{v}_k|\boldsymbol{0}, \boldsymbol{V}_k), \quad (34b)$$
$$\underline{\boldsymbol{y}}_k = \boldsymbol{h}_k(\underline{\boldsymbol{x}}_k) + \boldsymbol{G}_k\underline{\boldsymbol{n}}_k + \underline{\boldsymbol{e}}_k, \quad \underline{\boldsymbol{e}}_k \sim \mathcal{N}(\boldsymbol{e}_k|\boldsymbol{0}, \boldsymbol{R}_k), \quad (34c)$$

where the ambiguities evolve by a random walk with noise chosen by the cycle-slip detection in Sec. IV-C. The goal is to design an estimator based on the Bayesian interpretation in Sec. III-A, which if $\boldsymbol{f}_k(\boldsymbol{x}_k) = \boldsymbol{x}_k$, $\boldsymbol{h}_k(\boldsymbol{x}_k) = \boldsymbol{H}_k\boldsymbol{x}$, and $\boldsymbol{W}_k = \boldsymbol{0}$, $\boldsymbol{V}_k \to \infty$ equates to the constrained RMILS scheme in Sec. III, hence approximately solving (9).

Assume knowledge of a solution to the relaxed estimation problem at a time $k-1$ (i.e., with real-valued ambiguities), $p(\boldsymbol{x}_{k-1}^f, \boldsymbol{n}_{k-1}^f|\boldsymbol{Y}_{k-1})$, as an approximate Gaussian density,

$$\mathcal{N}\left(\begin{bmatrix}\boldsymbol{x}_{k-1}^f\\ \boldsymbol{n}_{k-1}^f\end{bmatrix}\middle|\begin{bmatrix}\boldsymbol{m}_{k-1|k-1}^{\boldsymbol{x}^f}\\ \boldsymbol{m}_{k-1|k-1}^{\boldsymbol{n}^f}\end{bmatrix}, \begin{bmatrix}\boldsymbol{\Sigma}_{k-1|k-1}^{\boldsymbol{x}^f\boldsymbol{x}^f} & \boldsymbol{\Sigma}_{k-1|k-1}^{\boldsymbol{x}^f\boldsymbol{n}^f}\\ \star & \boldsymbol{\Sigma}_{k-1|k-1}^{\boldsymbol{n}^f\boldsymbol{n}^f}\end{bmatrix}\right). \quad (35)$$

We can propagate (35) forward through the dynamics in (34a) yielding $p(\boldsymbol{x}_k^f, \boldsymbol{n}_k^f|\boldsymbol{Y}_{k-1})$, and compute the joint PDF of predicted measurement and states through (34c),

$$p(\boldsymbol{x}_k^f, \boldsymbol{n}_k^f, \boldsymbol{y}_k|\boldsymbol{Y}_{k-1}) =$$
$$\mathcal{N}\left(\begin{bmatrix}\boldsymbol{x}_k^f\\ \boldsymbol{n}_k^f\\ \boldsymbol{y}_k\end{bmatrix}\middle|\begin{bmatrix}\boldsymbol{m}_{k|k-1}^{\boldsymbol{x}^f}\\ \boldsymbol{m}_{k|k-1}^{\boldsymbol{n}^f}\\ \boldsymbol{m}_{k|k-1}^{\boldsymbol{y}}\end{bmatrix}, \begin{bmatrix}\boldsymbol{\Sigma}_{k|k-1}^{\boldsymbol{x}^f\boldsymbol{x}^f} & \boldsymbol{\Sigma}_{k|k-1}^{\boldsymbol{x}^f\boldsymbol{n}^f} & \boldsymbol{\Sigma}_{k|k-1}^{\boldsymbol{x}^f\boldsymbol{y}}\\ \star & \boldsymbol{\Sigma}_{k|k-1}^{\boldsymbol{n}^f\boldsymbol{n}^f} & \boldsymbol{\Sigma}_{k|k-1}^{\boldsymbol{n}^f\boldsymbol{y}}\\ \star & \star & \boldsymbol{\Sigma}_{k|k-1}^{\boldsymbol{y}\boldsymbol{y}}\end{bmatrix}\right), \quad (36)$$

for example, using the explicit linearization in Remark 2. From Remark 3, the moments in (36) can also be evaluated using statistical linearization in [11]. Conditioning on the measurement yields the relaxed solution at a time step k as an approximate Gaussian density with

$$p(\boldsymbol{x}_k^f, \boldsymbol{n}_k^f|\boldsymbol{Y}_k) = \mathcal{N}\left(\begin{bmatrix}\boldsymbol{x}_k^f\\ \boldsymbol{n}_k^f\end{bmatrix}\middle|\begin{bmatrix}\boldsymbol{m}_{k|k}^{\boldsymbol{x}^f}\\ \boldsymbol{m}_{k|k}^{\boldsymbol{n}^f}\end{bmatrix}, \begin{bmatrix}\boldsymbol{\Sigma}_{k|k}^{\boldsymbol{x}^f\boldsymbol{x}^f} & \boldsymbol{\Sigma}_{k|k}^{\boldsymbol{x}^f\boldsymbol{n}^f}\\ \star & \boldsymbol{\Sigma}_{k|k}^{\boldsymbol{n}^f\boldsymbol{n}^f}\end{bmatrix}\right).$$

We can then use the posterior in the relaxed estimation problem, to find the integer ambiguity hypotheses by solving the ILS problem similar to (13),

$$\boldsymbol{n}_k^I = \underset{\boldsymbol{n}_k\in\mathbb{Z}^n}{\arg\min}\|\boldsymbol{n}_k - \boldsymbol{m}_{k|k}^{\boldsymbol{n}^f}\|_{(\boldsymbol{\Sigma}_{k|k}^{\boldsymbol{n}^f\boldsymbol{n}^f})^{-1}}. \tag{37}$$

In this paper, we use the LAMBDA method [3] and the sequential bootstrapping method [1]. Based on the integer hypotheses, we form a constrained measurement by

$$\boldsymbol{z}_k = \boldsymbol{y}_k - \boldsymbol{G}_k\boldsymbol{n}_k^I \in \mathbb{R}^p. \tag{38}$$

With $p(\boldsymbol{x}_{k-1}|\boldsymbol{Z}_{k-1}) \approx \mathcal{N}(\boldsymbol{x}_{k-1}|\boldsymbol{m}_{k-1|k-1}^{\boldsymbol{x}}, \boldsymbol{\Sigma}_{k-1|k-1}^{\boldsymbol{xx}})$ and $\boldsymbol{m}_{k-1|k-1}^{\boldsymbol{x}}$, $\boldsymbol{\Sigma}_{k-1|k-1}^{\boldsymbol{xx}}$ known from the previous time-step, this estimate is propagated through the dynamics in (34a) to form the joint distribution between predicted state and predicted integer constrained measurement through (34c), as

$$p(\boldsymbol{x}_k, \boldsymbol{z}_k|\boldsymbol{Z}_{k-1}) \approx \left(\begin{bmatrix}\boldsymbol{x}_k\\ \boldsymbol{z}_k\end{bmatrix}\middle|\begin{bmatrix}\boldsymbol{m}_{k|k-1}^{\boldsymbol{x}}\\ \boldsymbol{m}_{k|k-1}^{\boldsymbol{z}}\end{bmatrix}, \begin{bmatrix}\boldsymbol{\Sigma}_{k|k-1}^{\boldsymbol{xx}} & \boldsymbol{\Sigma}_{k|k-1}^{\boldsymbol{xz}}\\ \star & \boldsymbol{\Sigma}_{k|k-1}^{\boldsymbol{zz}}\end{bmatrix}\right).$$

The moments of the joint distribution can again be computed by a statistical linearization, from Remark 3. Finally, through a Bayesian update, we evaluate $p(\boldsymbol{x}_k|\boldsymbol{Z}_k)$ and output $\boldsymbol{m}_{k|k}^{\boldsymbol{x}}$ and $\boldsymbol{\Sigma}_{k|k}^{\boldsymbol{xx}}$ along with the integer hypotheses $\boldsymbol{n}_k^I$. The method is summarized in Algorithm 1. The algorithm using Remark 2 is referred to as the MI-EKF, and using Remark 3 results in the MI-LRKF. Both algorithms become the constrained RMILS algorithm in the linear static setting if $\boldsymbol{V}_k \to \infty$ and $\boldsymbol{W}_k = \boldsymbol{0}$ for all k.

V. NUMERICAL EXAMPLES

Consider a single-band GNSS receiver whose motion is governed by a three-dimensional double integrator with a position $\boldsymbol{p}^R(t) \in \mathbb{R}^3$ and velocities $\boldsymbol{v}^R(t) \in \mathbb{R}^3$, driven by a random walk characterized by a variance $\alpha > 0$. We consider a total of N visible satellites, each of which is associated with several biases. The biases are collected in $\boldsymbol{\theta}(t) \in \mathbb{R}^{3N}$, and are subject to a random walk with variance $\beta > 0$. The real-valued state vector is $\boldsymbol{x}_k = (\boldsymbol{p}_k^{R\top}, \boldsymbol{v}_k^{R\top}, \boldsymbol{\theta}_k^\top)^\top$. Zero-order hold sampling with time-step h gives

$$\underline{\boldsymbol{x}}_{k+1} = \begin{bmatrix}\boldsymbol{I} & h\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}\underline{\boldsymbol{x}}_k + \underline{\boldsymbol{w}}_k, \quad \underline{\boldsymbol{w}}_k \sim \mathcal{N}(\boldsymbol{w}_k|\boldsymbol{0}, \boldsymbol{W}_k), \tag{39}$$

where

$$\boldsymbol{W}_k = \begin{bmatrix}(\alpha h^3/3)\boldsymbol{I}_3 & (\alpha h^2/2)\boldsymbol{I}_3 & \boldsymbol{0}\\ (\alpha h^2/2)\boldsymbol{I}_3 & \alpha h\boldsymbol{I}_3 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & h\beta\boldsymbol{I}_{3N}\end{bmatrix}. \tag{40}$$

Algorithm 1: MI-EKF/LRKF, with adaptive priors.

Receive $\boldsymbol{m}_{0|0}^{\boldsymbol{x}^f}, \boldsymbol{m}_{0|0}^{\boldsymbol{n}^f}, \boldsymbol{\Sigma}_{0|0}^{\boldsymbol{x}^f\boldsymbol{x}^f}, \boldsymbol{\Sigma}_{0|0}^{\boldsymbol{x}^f\boldsymbol{n}^f}, \boldsymbol{\Sigma}_{0|0}^{\boldsymbol{n}^f\boldsymbol{n}^f}, \boldsymbol{m}_{0|0}^{\boldsymbol{x}}, \boldsymbol{\Sigma}_{0|0}^{\boldsymbol{xx}}$
for $k = 1$ **to** K **do**
  Receive $\boldsymbol{y}_k$
  // Compute ambiguity prior
  Compute $\boldsymbol{c}_k$ using (32) and $\boldsymbol{V}_{k-1}$ using (33)
  // Compute the unconstrained joint
  Evaluate $p(\boldsymbol{x}_k^f, \boldsymbol{n}_k^f, \boldsymbol{y}_k|\boldsymbol{Y}_{k-1})$ by Remark 2 or 3
  // Condition on measurement
  Evaluate $p(\boldsymbol{x}_k^f, \boldsymbol{n}_k^f|\boldsymbol{Y}_k)$
  // Compute integer ambiguity hypotheses
  $\boldsymbol{n}_k^I = \underset{\boldsymbol{n}_k\in\mathbb{Z}^n}{\arg\min}\|\boldsymbol{n}_k - \boldsymbol{m}_{k|k}^{\boldsymbol{n}^f}\|_{(\boldsymbol{\Sigma}_{k|k}^{\boldsymbol{n}^f\boldsymbol{n}^f})^{-1}}$
  // Compute constrained measurement
  $\boldsymbol{z}_k = \boldsymbol{y}_k - \boldsymbol{G}_k\boldsymbol{n}_k^I$
  // Compute constrained joint
  Evaluate $p(\boldsymbol{x}_k, \boldsymbol{z}_k|\boldsymbol{Z}_{k-1})$ by Remark 2 or 3
  // Condition on the constrained measurement
  Evaluate $p(\boldsymbol{x}_k|\boldsymbol{Z}_k)$
  Output $\hat{\boldsymbol{x}}_k = \boldsymbol{m}_{k|k}^{\boldsymbol{x}}$ and $\boldsymbol{n}_k^I$
end

In addition to $\boldsymbol{x}_k$, we have a total of $M = N-1$ ambiguities denoted $\boldsymbol{n}_k \in \mathbb{Z}^M$, which evolve by an multivariate integer jump process according to (29), here characterized by $\boldsymbol{a} = 10 \cdot \boldsymbol{1}_M$ and the jump probabilities $\boldsymbol{b} = (0.01h) \cdot \boldsymbol{1}_M$.

The satellite positions are $\boldsymbol{p}^i \in \mathbb{R}^3$, corresponding to the GPS satellite system, and we assume a known static base station located at $\boldsymbol{p}^B \in \mathbb{R}^3$ and 10^4 [m] from the initial receiver position $\boldsymbol{p}_0^R$, both on the Earth's surface. We define the Euclidean distance between satellite and receiver, the single difference and double-difference operators as

$$\rho_{R,k}^i = \|\boldsymbol{p}_k^R - \boldsymbol{p}_k^i\|_2, \tag{41a}$$

$$\Delta_{BR}^i(\rho_k) = \rho_{R,k}^i - \rho_{B,k}^i, \tag{41b}$$

$$\nabla\Delta_{BR}^{ij}(\rho_k) = \Delta_{BR}^i(\rho_k) - \Delta_{BR}^j(\rho_k). \tag{41c}$$

Let $\boldsymbol{\rho}_k = (\Delta_{BR}^1(\rho_k), ..., \Delta_{BR}^N(\rho_k))^\top$ and define $\boldsymbol{M} = \begin{bmatrix}\boldsymbol{1} & -\boldsymbol{I}\end{bmatrix}$. Hence, the double-difference measurements are

$$\underline{\boldsymbol{y}}_k^P = \boldsymbol{M}\underline{\boldsymbol{\rho}}_k + \boldsymbol{M}\underline{\boldsymbol{\theta}}_k^I + \boldsymbol{M}\underline{\boldsymbol{\theta}}_k^P + \underline{\boldsymbol{\epsilon}}_k, \tag{42a}$$

$$\underline{\boldsymbol{y}}_k^\Phi = \boldsymbol{M}\underline{\boldsymbol{\rho}}_k + \lambda\underline{\boldsymbol{n}}_k - \boldsymbol{M}\underline{\boldsymbol{\theta}}_k^I + \boldsymbol{M}\underline{\boldsymbol{\theta}}_k^\Phi + \underline{\boldsymbol{\eta}}_k, \tag{42b}$$

with zero-mean Gaussian noise terms $\boldsymbol{\eta}_k$ and $\boldsymbol{\epsilon}_k$. We estimate the kinematic states $\boldsymbol{p}_k^R$ and $\boldsymbol{v}_k^R$; the double difference integer ambiguities, $\boldsymbol{n}_k \in \mathbb{Z}^M$; the single difference ionospheric biases, $\boldsymbol{\theta}_k^I \in \mathbb{R}^N$; the single difference code correction terms, $\boldsymbol{\theta}_k^P \in \mathbb{R}^N$; and the single difference phase correction terms, $\boldsymbol{\theta}_k^\Phi \in \mathbb{R}^N$. The states are not fully observable in (42a) and (42b), and the reason for including the biases in this manner is to facilitate the inclusion of satellite dependent correction terms and modeled ionospheric biases (e.g., using a slanted total electron count grid interpolation [8]), introduced as a measurement on the associated bias. To facilitate this, we model three additional measurement equations,

$$\underline{\boldsymbol{y}}_k^I = \underline{\boldsymbol{\theta}}_k^I + \underline{\boldsymbol{r}}_k^I, \quad \underline{\boldsymbol{y}}_k^P = \underline{\boldsymbol{\theta}}_k^P + \underline{\boldsymbol{r}}_k^P, \quad \underline{\boldsymbol{y}}_k^\Phi = \underline{\boldsymbol{\theta}}_k^\Phi + \underline{\boldsymbol{r}}_k^\Phi. \tag{42c}$$

TABLE I

NOMINAL PARAMETERS USED IN THE SIMULATION

Parameter	Value	Parameter	Value
h	0.1	σ_{slip}	10
α	0.01	σ_{stay}	0.05
β	0.01	σ_P	0.5
a	10	σ_Φ	0.01
b	0.05	σ_θ	0.01
λ	0.2	N	13

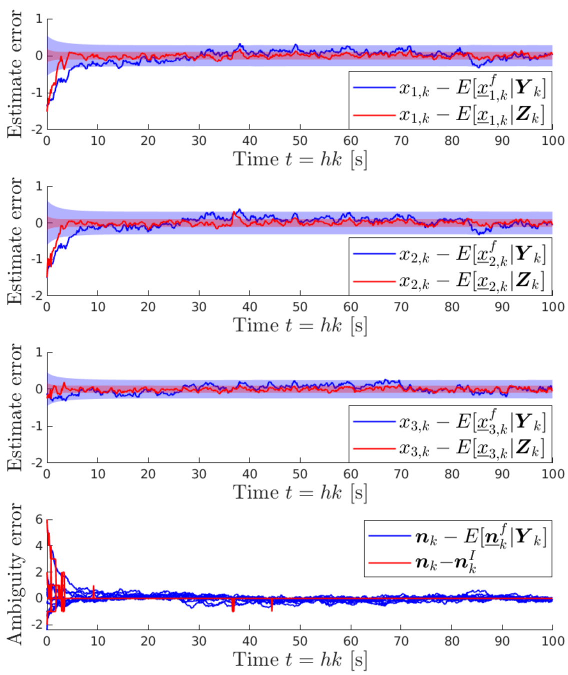

Fig. 1. *Top three subplots*: Mean positional estimate errors in the relaxed solution (blue) and the integer constrained solution (red) when running the MI-LRKF with a Spherical Cubature point set and adaptive ambiguity priors, with an estimated 2σ-confidence interval. *Bottom*: Mean ambiguity estimate error in the relaxed estimation problem and fixed integer hypothesis.

Collecting $\boldsymbol{e}_k = (\boldsymbol{\epsilon}_k^\top, \boldsymbol{\eta}_k^\top, \boldsymbol{r}_k^I, \boldsymbol{r}_k^P, \boldsymbol{r}_k^\Phi)^\top$, we assume that

$$\underline{\boldsymbol{e}}_k \sim \mathcal{N}\left(\boldsymbol{e}_k \middle| \mathbf{0}, \begin{bmatrix} 2\sigma_P^2 \boldsymbol{M}\boldsymbol{M}^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\sigma_\Phi^2 \boldsymbol{M}\boldsymbol{M}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma_\theta^2 \boldsymbol{I}_{3N} \end{bmatrix}\right),$$

as is common in prior works [2], resulting in a model in (34), with nominal parameters in Table I. Given (42), the cycle-slip detection in (31) is well defined at all times. The synthetic data are generated from the model in (34), but with the integer ambiguity biases evolving in time by (29).

The resulting estimates of the MI-LRKF, with adaptive ambiguity priors in Algorithm 1, using a point set corresponding to the Spherical Cubature rule used to define the CKF in [16] is shown in Fig. 1 in terms of the positional estimate errors and their variances in the relaxed solution (blue) and the constrained solution (red), as well as the mean ambiguity estimate errors in the relaxed estimate density (blue) and the fixed integer hypotheses (red).

Fixing the integer ambiguity biases using the dual density MI-LRKFs outlined in Algorithm 1 with the adaptive ambiguity priors is a viable approach for the GNSS-positioning problem. When studying the errors in Fig. 1, both the estimate variance seems consistent with the mean error both in the relaxed and constrained solutions, and the 12-dimensional ambiguity hypotheses vector is estimated correctly after the initial transient with the exception of deviations in single dimensions occurring at three distinct time-steps, despite the frequent occurrence of cycle slips in the synthetic data.

VI. CONCLUSIONS

We considered the GNSS positioning problem and devised recursive filters equating to the constrained and approximately optimal solution to a maximum-likelihood problem in the linear setting. We relaxed the original optimization problem and considered the simultaneous propagation of two separate densities over the real-valued states, one conditioned on all prior measurements, and another conditioned on all prior measurements and fixed integer ambiguity hypotheses. We have (i) shown how to replace the explicit linearization common in GNSS positioning with statistical linearization in [11]. We have also (ii) devised a method of adapting the prior of the relaxed ambiguity distribution based on a modeled underlying integer jump process. We demonstrated the approach in a filtering setup with synthetic data.

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