Robust Adaptive Dynamic Mode Decomposition for Reduce Order Modelling of Partial Differential Equations

Kalur, Aniketh; Nabi, Saleh; Benosman, Mouhacine

TR2021-059   June 04, 2021

Abstract

This work focuses on the design of stable reduced-order models (ROMs) for partial differential equations (PDEs) with parametric uncertainties. More specifically, we focus here on using dynamic mode decomposition (DMD) to reduce a PDE to a DMD-ROM and then pose the ROM stabilization or closure problem in the framework of nonlinear robust control. Using this robust control framework, we design two DMD-ROM closure models that are robust to parametric uncertainties and truncation of modes. We finally add an adaptation layer to our framework, where we tune the closure models in real-time, using data-driven extremum seeking controllers.

American Control Conference (ACC)

Robust Adaptive Dynamic Mode Decomposition for Reduce Order Modelling of Partial Differential Equations

Aniketh Kalur[1], Saleh Nabi[2], and Mouhacine Benosman[2]

***Abstract*— This work focuses on the design of stable reduced-order models (ROMs) for partial differential equations (PDEs) with parametric uncertainties. More specifically, we focus here on using dynamic mode decomposition (DMD) to reduce a PDE to a DMD-ROM and then pose the ROM stabilization or closure problem in the framework of nonlinear robust control. Using this robust control framework, we design two DMD-ROM closure models that are robust to parametric uncertainties and truncation of modes. We finally add an adaptation layer to our framework, where we tune the closure models in real-time, using data-driven extremum seeking controllers.**

I. Introduction

The use of reduced-order models (ROMs), i.e., reducing partial differential equations (PDEs) model to a system of finite-dimensional ordinary differential equations (ODEs), in control and optimization has led to practical solutions for extremely challenging problems, such as control of thermo-fluidic systems [1], [2], windfarms [3], and solutions to the Hamilton–Jacobi–Bellman equation arising in nonlinear feedback control [4], among others. The presence of increasingly large data sets, from experiments or simulations, enables the design of ROMs using methods like proper orthogonal decomposition (POD) or dynamic mode decomposition (DMD), both of which can extract tractable and physically relevant information from the data at a given set of the system's parameters. However, one major challenge is that ROMs can introduce stability loss and prediction degradation. These degradations are mainly due to the truncation of higher modes and parametric uncertainties. More specifically, the basis functions (spatial modes) obtained from data snapshots at one given set of parameters, may show deterioration in the accuracy of the ROMs prediction or even become unstable when applied to represent the solutions for a different range of parameters.

Many methods have been developed to remedy such difficulties, aiming at what is known as stable model reduction. This paper investigates the solution to such issues, especially for the DMD method, which, unlike POD, allows the selected basis to be directly associated with desired characteristic frequencies and growth/decay rates. Furthermore, as revealed by [5], DMD has a strong connection to Koopman operator theory [6], which makes it more appealing for predicting and analyzing nonlinear dynamical systems.

For the purpose of robust stable model reduction, we focus on the so-called *closure models*. Indeed, various approaches are reported in the literature to determine closure models, e.g. variational data assimilation [7], physics-inspired closure models [8], and Lyapunov-based stable model reduction [9], [10]. We propose to formulate the closure model problem as a virtual robust control (stabilization and tracking) problem, and then, using nonlinear robust control theory, we propose two robustly stabilizing closure models. We propose using a learning-based method, based on multi-parametric extremum seeking (MES) control, to automatically tune the coefficients of the closure models to optimally track the actual PDE model solutions. As a test problem, we consider the Burgers equation, with uncertain Reynolds number (viscosity coefficient).

The rest of the paper is organized as follows: Section II establishes some basic notations and definitions. Section III introduces the concept of DMD. Section IV represents the main contribution of the paper, consisting of two new robust DMD-ROM closure models, and their MES-based auto-tuning method. In Section V we present the test case and validation. Finally, conclusions and future steps are discussed in Section VI.

II. Basic Notation and Definitions

For a vector $\mathbf{x} \in \mathbb{R}^n$, the transpose is denoted by $\mathbf{x}^T$. The Euclidean vector norm for $\mathbf{x} \in \mathbb{R}^n$ is denoted by $\| \cdot \|$ so that $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$. $\|M\|_F$ denotes the Frobenius norm of the matrix $M \in \mathrm{R}^{r\times r}$. We shall abbreviate the time derivative by $\dot{f}(t,\mathbf{x}) = \frac{\partial}{\partial t} f(t,\mathbf{x})$, and consider the following Hilbert spaces: $\mathcal{H} = L^2(\Omega)$, $\Omega = (0,1)$, which is the space of Lebesgue square integrable functions, i.e., $f \in \mathcal{H}$, iff $\int_\Omega |f(x)|^2 dx < \infty$. We define the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and the associated norm $\| \cdot \|_{\mathcal{H}}$ on $\mathcal{H}$ as $\langle f, g\rangle_{\mathcal{H}} = \int_\Omega f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}$, for $f, g \in \mathcal{H}$, and $\|f\|^2_{\mathcal{H}} = \int_\Omega |f(\mathbf{x})|^2 d\mathbf{x}$. A function $f(t,\mathbf{x})$ is in $L^2([0,t_f];\mathcal{H})$ if for each $0 \le t \le t_f$, $f(t,\cdot) \in \mathcal{H}$, and $\int_0^{t_f} \|f(t,\cdot)\|^2_{\mathcal{H}} dt < \infty$.

[1] Department of Aerospace Engineering and Mechanics, Univ. of Minnesota. This work was completed during an internship at MERL.
[2] Mitsubishi Electric Research Laborotries (MERL), Cambridge, MA, 02139, USA

III. Data-Driven Modeling: Dynamic Mode Decomposition

Dynamic mode decomposition (DMD) is a data-driven method to uncover the underlying dynamics of a system from data. In this paper, we typically consider data collected from a dynamical system of the form

$$\dot{\mathbf{z}}(t) = \mathcal{F}(\mathbf{z}(t), \nu), \qquad \mathbf{z}(0) \in \mathcal{Z}, \tag{1}$$

where $\mathcal{Z}$ is an infinite-dimensional Hilbert space, and where ν represents a parameter of the system.

Assumption 1: The solutions of the original PDE model (1) are assumed to be in $L^2([0,\infty); \mathcal{Z})$, $\forall \nu \in \mathbb{R}$.

Solutions to the PDE model (1) can be approximated in a finite dimensional subspace $\mathcal{Z}^n \subset \mathcal{Z}$ through expensive numerical discretization, which can be impractical for real-time applications. In many systems solutions of the PDE may be well-approximated using only a few basis functions [11]. In this paper we use DMD to construct these model reduction basis functions.

DMD is a data-driven technique that has been widely used in the fluid dynamics community to extract spatio-temporal modes from complex and dynamically evolving data-sets [12], [13]. DMD operates on empirical snapshot data to extract rich dynamical information that can be used to extract underlying dynamics from data. We consider a time series of data, collected at various instances in time, where the time is presented by t_k, where k is the time index. The data at the t_k and t_{k+1} time instance are given by vector $\mathbf{x}_k \in \mathbb{R}^n$ and $\mathbf{x}_{k+1} \in \mathbb{R}^n$ both of which are subsets of $\mathcal{Z}^n$.

Once we collect measurement data from the system, the time-series data is stored in a snapshot matrix $X = \begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{m-1}\end{bmatrix} \in \mathbb{R}^{n\times m}$ and a time shifted snapshot matrix $Y = \begin{bmatrix}\mathbf{x}_2 & \mathbf{x}_3 & \cdots & \mathbf{x}_m\end{bmatrix} \in \mathbb{R}^{n\times m}$. We seek to find a linear map $A_{orig} \in \mathbb{R}^{n\times n}$ such that $Y = A_{orig}X$. Using singular value decomposition (SVD), $X = U\Sigma V^*$, then a lower-dimensional proxy system $A = U^* A_{orig} U$ can be formed, where U contains the leading r left singular vectors of X and the reduced-rank linear operator is given by

$$A = U^* Y V \Sigma^{-1}, \tag{2}$$

where, $A \in \mathbb{R}^{r\times r}$ is the reduced-rank operator of A_{orig}.

IV. Closure models and Extremum Seeking Based Tuning

The reduced-order model obtained by DMD is

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{3}$$

where $A \in \mathrm{R}^{r\times r}$ is the uncertain state matrix. Furthermore, to retain the physical nature of the diffusive PDE, we explicitly add a diffusive term to the ROM (3), as

$$\dot{\mathbf{x}} = A\mathbf{x} - \nu_1 D\mathbf{x}, \tag{4}$$

where $D > 0$ represents a constant viscosity damping matrix, and $\nu_1 > 0$ a viscosity gain which will be used later to auto-tune the reduced order model tracking performance. This explicit addition of diffusion terms in the ROM is common in ROMs literature, e.g., ([10] and references therein).

In this work, to account for physical model parametric uncertainties e.g., Reynolds number mismatch in wind-farm applications [14], Richardson number uncertainties in HVAC applications [15], as well as, uncertainties induced by modal or basis function truncation, we formulate the DMD-ROM construction as a robust control problem, as

$$\dot{\mathbf{x}} = (A_n + \Delta A)\mathbf{x} - \nu_1(D_n + \Delta D)\mathbf{x} + \mathbf{u}, \tag{5}$$

where, $D_n > 0, \Delta D > 0$, A_n and the uncertainty terms ΔA, ΔD satisfy the following assumptions.

Assumption 2: The nominal state matrix A_n is stable, i.e., $\lambda(A_n) < 0$.

Assumption 3: The state matrix uncertainty term ΔA is bounded, such that $\|\Delta A\|_F \leq \overline{\Delta A}$.

Assumption 4: The viscosity damping matrix uncertainty term ΔD is bounded, such that $\|\Delta D\|_F \leq \overline{\Delta D}$.

The virtual control term $\mathbf{u}$ is added here to represent a general closure model term, added to stabilize the DMD-ROM model. The difference with existing physics based literature on closure models, e.g., [8] is that we are formulating the closure model problem, in this context of DMD-ROM, as a robust stabilization problem.

We propose two closure models which we call, robust correction and robust correction with vanishing viscosity, respectively. We introduce these closure models, analyze their stability and robustness using Lyapunov theory in the follow section.

A. Closure models for DMD-ROM

1) Closure model 1-Robust correction: First, we consider the nonlinear closure model $\mathbf{u} = \mathbf{u}^1_{cl}(\mathbf{x})$, such that:

$$\dot{\mathbf{x}} = (A_n + \Delta A)\mathbf{x} - \nu_1(D_n + \Delta D)\mathbf{x} + \mathbf{u}^1_{cl}(\mathbf{x}) \tag{6}$$

where,

$$\mathbf{u}^1_{cl}(\mathbf{x}) = -\nu_2\left(\overline{\Delta A}\|\mathbf{x}\| - \nu_1\overline{\Delta D}\|\mathbf{x}\|\right) D_n\mathbf{x},\ \nu_2 > 0. \tag{7}$$

The stability property of this closure model is summarized in the following Lemma.

Lemma 1: The solutions of the DMD-ROM (6), under Assumptions 2-4, with the closure model (7), are bounded and converge to the positive invariant set S, given by

$$S = \{\mathbf{x} \in \mathbb{R}^r : 1 - \nu_2\lambda_{max}(D_nP)\|\mathbf{x}\| ... \\ ... - \nu_1 \frac{\lambda_{max}(D_nP)}{\overline{\Delta A}\|P\|_F - \nu_1\overline{\Delta D}\|P\|_F)} \geq 0\} \tag{8}$$

Proof: To analyze stability of this correction term, we define a candidate Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$ such that $V(\mathbf{x}) > 0,\ \forall \mathbf{x} \neq \mathbf{0}$ and $P = P^T > 0$. The time derivative of $V(\mathbf{x})$ along all trajectories is given by:

$$\dot{V}(\mathbf{x}) = \frac{1}{2}\left(\dot{\mathbf{x}}^T P \mathbf{x} + \mathbf{x}^T P \dot{\mathbf{x}}\right). \tag{9}$$

Substituting Eq. (6) into Eq. (9), we get,

$$\dot{V}(\mathbf{x}) = \frac{1}{2}\left[(A_n + \Delta A)\mathbf{x} - \nu_1 (D_n + \Delta D)\mathbf{x} + \mathbf{u}^1_{cl}(\mathbf{x})\right]^T P \mathbf{x} \cdots + \frac{1}{2}\mathbf{x}^T P \left[(A_n + \Delta A)\mathbf{x} - \nu_1 (D_n + \Delta D)\mathbf{x} + \mathbf{u}^1_{cl}(\mathbf{x})\right] \tag{10}$$

therefore,

$$\dot{V}(\mathbf{x}) = \frac{1}{2}\left\{\mathbf{x}^T(A_n^T P + P A_n)\mathbf{x} + \mathbf{x}^T(\Delta A^T P + P\Delta A)\mathbf{x} ... -\nu_1 \mathbf{x}^T(D_n^T P + P D_n)\mathbf{x} - \nu_1 \mathbf{x}^T(\Delta D^T P + P\Delta D)\mathbf{x} ... + \mathbf{u}^1_{cl}(\mathbf{x})^T P \mathbf{x} + \mathbf{x}^T P \mathbf{u}^1_{cl}(\mathbf{x})\right\}. \tag{11}$$

In Eq. (10), we know that $A_n^T P + P A_n = -Q$, since the nominal system is stable, here $Q > 0$. This gives us the relation,

$$\dot{V}(\mathbf{x}) \leq \frac{1}{2}\left\{\mathbf{x}^T(\Delta A^T P + P\Delta A)\,\mathbf{x} - ... ...\nu_1 \mathbf{x}^T(D_n^T P + P D_n)\mathbf{x} - \nu_1 \mathbf{x}^T(\Delta D^T P + P\Delta D)\mathbf{x} ... ... + \mathbf{u}^1_{cl}(\mathbf{x})^T P \mathbf{x} + \mathbf{x}^T P \mathbf{u}^1_{cl}(\mathbf{x})\right\} \tag{12}$$

The upper bound of the time derivative of Lyapunov function is given as

$$\dot{V}(\mathbf{x}) \leq \|\mathbf{x}\|^2\left(\overline{\Delta A}\|P\|_F\right) - \nu_1\|\mathbf{x}\|^2\left(\lambda_{max}(D_n P)\right)\cdots - \nu_1\|\mathbf{x}\|^2\left(\overline{\Delta D}\|P\|_F\right)\cdots - \nu_2\|\mathbf{x}\|^2\left(\overline{\Delta A}\|P\|_F - \nu_1\overline{\Delta D}\|P\|_F\right)\|\mathbf{x}\|\lambda_{max}(D_n) \tag{13}$$

this leads to,

$$\dot{V}(\mathbf{x}) \leq \|\mathbf{x}\|^2(\overline{\Delta A}\|P\|_F - \nu_1\overline{\Delta D}\|P\|_F)\cdots \left[1 - \nu_2\lambda_{max}(D_n)\|\mathbf{x}\| - \nu_1\frac{\lambda_{max}(D_n P)}{\overline{\Delta A}\|P\|_F - \nu_1\overline{\Delta D}\|P\|_F)}\right], \tag{14}$$

which shows the convergence of solutions to the invariant set S.

2) Closure model 2- Robust correction with vanishing viscosity: In addition to the closure term shown in above, we add a time varying exponential decaying term to ensure faster convergence. The time varying closure model is given us:

$$\dot{\mathbf{x}} = (A_n + \Delta A)\mathbf{x} - \nu_1(D_n + \Delta D)\mathbf{x} + \mathbf{u}^2_{cl}(\mathbf{x}) \tag{15}$$

where,

$$\mathbf{u}^2_{cl}(\mathbf{x}) = -\nu_2(\overline{\Delta A} + \overline{\Delta D})\|\mathbf{x}\|D_n\mathbf{x} - \nu_1 e^{-\alpha t}D_n\mathbf{x} \tag{16}$$

The stability analysis of the DMD-ROM (15), with the closure model (16) is given below.

***Lemma** 2:* The solutions of the DMD-ROM (15), under Assumptions 2-4, with the closure model (16), are bounded and converge to the positive invariant set S, given by

$$S = \{\mathbf{x} \in \mathbb{R}^r : 1 - \nu_2\lambda_{max}(D_n P)\|\mathbf{x}\| ... ... - \nu_1\frac{\lambda_{max}(D_n P)}{\overline{\Delta A}\|P\|_F - \nu_1\overline{\Delta D}\|P\|_F)} \geq 0\}. \tag{17}$$

Furthermore, the convergence to S is accelerated proportionally to $\nu_1 e^{-\alpha t}\lambda_{max}(D_n P)$.

Proof: The proof follows similar steps as in the proof of Lemma 1.

B. Extremum seeking-based closure models adaptive tuning

Multi-parametric extremum seeking (MES) is a model free control algorithm, often used to optimize a given performance cost without closed-form information on the cost, e.g., [16]. However, MES control can also be used for open-loop model parametric identification, and feedback gain tuning, e.g., [17], [18]. We follow similar ideas here, and propose to use MES to auto-tune the closure models described in Eq. (6) and Eq. (15), by continually updating the parameter weights ν_1 and ν_2, from online measurements from the system.

In this section, we will briefly describe the MES algorithm used to update the closure model parameters ν_1 and ν_2, in an online setting. We first define a suitable learning cost function for the MES algorithm. The learning cost function is a positive definite function of the norm of error between the measured output of the full-order model (FOM) (real system), and the output from the DMD-ROM with corrections in Eq. (6), Eq. (15), respectively. Given the output $\mathbf{y}(t) = C\mathbf{x}(t)$, where C is the output matrix, and $\mathbf{x}(t)$ is the state of the system at time t. The cost function (J) is given as

$$J(\nu_1, \nu_2) = \int_0^{t_f} \|\mathbf{y}(t) - \mathbf{y}_{ROM}(t, \nu_1, \nu_2)\|^2 dt. \tag{18}$$

Here, $t_f > 0$ denotes the learning time horizon, $\mathbf{y}_{ROM}$ corresponds to the output of the DMD-ROM with closure model. In this work the error is computed online using measurements from exact solutions of the PDE, however, the same error can be computed in real applications, by direct measurements of the system.

We list some classical assumptions on the learning cost functions, which are needed to ensure some MES convergence guarantees.

***Assumption** 5:* The cost function J has a local minima at $\boldsymbol{\nu}^{opt} = (\nu_1^{opt}, \nu_2^{opt})$.

***Assumption** 6:* The cost function J is analytic and its variation with respect to the parameter $\boldsymbol{\nu} = [\nu_1, \nu_2]$ is bounded in the neighborhood of the local minima $\boldsymbol{\nu}^{opt}$.

For simplicity of the presentation we consider a simple dither-based MES algorithm, given by:

$$\dot{z}_1(t) = a_1 sin(\omega_1 t + \frac{\pi}{2})J \tag{19}$$

$$\nu_1 = z_1 + a_1 sin(\omega_1 t - \frac{\pi}{2}) \tag{20}$$

$$\dot{z}_2(t) = a_2 sin(\omega_2 t + \frac{\pi}{2})J \tag{21}$$

$$\nu_2 = z_2 + a_2 sin(\omega_2 t - \frac{\pi}{2}) \tag{22}$$

The convergence properties of this algorithm are summarized in the following Lemma.

***Lemma** 3:* Consider the PDE (1) under Assumption 1, together with its DMD-ROM model (6), (7), or (15), (16). Furthermore, suppose the closure model coefficients $\nu_1,\ \nu_2$ are tuned using the MES algorithm (19)-(22) where $\omega_{\max} = \max(\omega_1, \omega_2) > \omega^{\text{opt}}$, ω^{opt} large enough, and $J(\cdot)$ is given by (18). Let $e_{\boldsymbol{\nu}}(t) := (\nu_1 - \nu_1^{opt},\ \nu_2 - \nu_2^{opt})$, then, under Assumptions 5, and 6, the norm of the distance to the optimal values admits the following bound

$$\|e_{\boldsymbol{\nu}}(t)\| \leq \frac{\xi_1}{\omega_{\max}} + \sqrt{a_1^2 + a_2^2},\ t \to \infty, \tag{23}$$

where $a_1,\ a_2 > 0,\ \xi_1 > 0$, and the learning cost function approaches its optimal value within the following upper-bound

$$\|J(\boldsymbol{\nu}) - J(\boldsymbol{\nu}^{\text{opt}})\| \leq \xi_2(\frac{\xi_1}{\omega} + \sqrt{a_1^2 + a_2^2}), \tag{24}$$

as $t \to \infty$, where $\xi_2 = \max_{\boldsymbol{\nu} \in \mathcal{N}(\boldsymbol{\nu}^{\text{opt}})} \|\nabla_{\boldsymbol{\nu}} J(\boldsymbol{\nu})\|$.
Proof: Refer to [18].

V. TEST-CASE AND NUMERICAL RESULTS

In this section, we discuss the results obtained by evaluating the proposed framework on the Burgers equation. Firstly, we briefly introduce the Burgers equation. Secondly, we describe the problem setup. Lastly, we investigate the performance of the tunable closure models for DMD-ROMs, by evaluating the error between the DMD-ROMS with closure terms and the standard DMD-based models.

A. Test-case: Burgers Equation

The proposed framework is tested on the viscous Burgers equation which describes the dynamics of a single advecting-diffusing wave in one dimension, given by

$$\frac{\partial \mathbf{v}}{\partial t} = \nu \frac{\partial^2 \mathbf{v}}{\partial \mathbf{x}^2} - \mathbf{v}\frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \tag{25}$$

where, $\mathbf{v}(t, \mathbf{x})$ is the velocity of the wave, $\mathbf{x}$ is the spatial direction along which the wave propagates, t is the time and ν is the viscosity of the wave.

Spectral methods are used to solve the Burgers equation by discretizing the $\mathbf{x}$ direction using Fourier series

$$\mathbf{v}(t, \mathbf{x}) = \sum_{k=-N}^{N} \hat{\mathbf{v}}_k e^{ik\mathbf{x}}. \tag{26}$$

Here, $\hat{\mathbf{v}}(t)$ is the Fourier coefficient, k is the wavenumber, and N is some finite value truncation of the wavenumber. By substituting Eq. (26) into Eq. (25), we obtain the following ordinary differential equation (ODE)

$$\dot{\hat{\mathbf{v}}} = -\nu k^2 \hat{\mathbf{v}} - ik\hat{\mathbf{v}}^2. \tag{27}$$

The above equation can now be solved for various wave numbers using standard solvers like ODE45 in MATLAB. The spatial direction x is discretized using 256 linearly spaced points and the system is initialized with an velocity profile given by $\mathbf{v}(t_0) = e^{-(\mathbf{x}+2)^2}$.

B. Problem Setup

In Eq (27), it is known that $\nu \propto \frac{1}{Re}$, where Re is Reynolds number. Therefore, changes in ν results directly in change of the flow regime the system is operating in. This variation in ν over time can causes many challenges while developing ROMs, since the ROMs need to be able to accommodate for parametric variations. Hence, we test the proposed frameworks and its efficacy in adapting to the ν parameter variations.

We first develop a ROM using DMD only (without closure model), this ROM is developed on data obtained from the Burgers equation at a nominal parameter $\nu_{\text{nom}} = 0.01$, using 50 time units worth of data. The domain is 1-dimensional in $\mathbf{x} \in [0\ 1]$ with periodic boundary conditions such that $\mathbf{v}(0) = \mathbf{v}(1)$. The resulting ROM developed on the nominal parameter has a reduced state size $r = 14$, while the original system is of size $n = 256$. However, due to disturbances and changes in the environment, we assume that the system is operating at the true value $\nu = 1$, which is far away from the nominal value (ν_{nom}). This discrepancy in parameter between the nominal value and the true value makes the ROM developed on the nominal value not usable, since the large variance between parameters ν_{nom} and ν may lead to an unstable ROM or may lead to large solution errors or both. In our study, with a DMD-ROM developed on ν_{nom} with size $r = 14$, we observe that the DMD-ROM is not stable due to presence of two positive eigenvalues $\lambda_1 = 0.00082 + i0.0577$ and $\lambda_2 = 0.0073 - i0.0596$ respectively. These unstable modes with low-energy cause the large error with oscillations (see orange curve in Fig. 2), which will eventually become radially unbounded.

C. Results

1) Results from robust correction: The difference in parameters ν_{nom} and ν creates a large discrepancy, which directly effects the performance of the ROM. Hence, we test the performance of the DMD-ROM with closure models (6), (7), and the tuning algorithm (19)-(22), with the constants: $a_1 = 10^{-3}$, $\omega_1 = 10, \phi_1 = \frac{\pi}{2}$ and $a_2 = 10^{-6}$, $\omega_2 = 1000$, $\phi_2 = \frac{\pi}{2}$, respectively. Using measurement data over 50 time units, we first report the convergence of the learning cost in Fig. 1(a). The values of ν_1 and ν_2 over learning iterations are shown in Fig. 1(b), where the final value is $\nu_1 = 0.043$ and $\nu_2 = -0.6 \times 10^{-6}$. We underline here that the optimal value of ν_2 is very small, which makes brute force manual tuning intractable. It should be noted that we stop the learning of parameters $\nu_{1,2}$ after 3000 iterations since the cost function has converged, any additional gains in $\nu_{1,2}$ will have only marginal effects in the results. The obtained values of ν_1 and ν_2 from MES algorithm are then tested to simulate the robust closure model for an extended time using measurement data elapsing 200 time units. Even though the parameters $\nu_{1,2}$ are learned over 50 time units worth of data, they perform well when applied to an extended time study with 200 time units worth of data, which shows the extrapolation performance of the obtained robust DMD-ROM models. We also compare the performance of the DMD-ROM with robust correction to that of the standard DMD. This result is shown by the blue curve in Fig. 2 , where it can be seen that the DMD-ROM with robust correction (blue) improves to model estimation error in comparison to the standard DMD (orange).

2) Results from Robust correction with vanishing viscosity: We report in this section the results corresponding to the robust correction with vanishing viscosity for the same problem. We observe the convergence of the cost function similar to that of Fig. 1(a).

The parameters ν_1 and ν_2 obtained from MES for 50 time units of data, are then applied on the system with robust correction with vanishing viscosity for an extended time of 200 time units. It can be seen from Fig. 2 that the DMD-ROM with vanishing viscosity term (green) converges faster than the DMD-ROM with robust correction (blue). This result is expected from the analysis results of Lemma 2. However, it should be noted that both correction methods do better than the standard DMD without any correction (orange in Fig.2).

3) Robustness to modes truncation: As mentioned in Section. V-B, we use a reduced-order model of size $r = 14$. To further investigate the robustness of the proposed DMD-ROMs, we study the correction terms obtained on the ROM model with $r = 14$ states and implement the same correction term on a ROM model system with $r = 5$ states. More specifically, the parameters ν_1 and ν_2 are identified on a model of size $r = 14$, then we use the obtained parameters on a model of reduced size $r = 5$. We aim to study the effect of uncertainty caused by error in modes truncation. To this effect, as shown in Fig. 3, we see that the robust correction terms effectively minimize the model estimation error. This is despite the fact that the DMD-ROM without closure models is unstable.

In Fig. 3, we can see that the correction term is robust to uncertainty due to modes truncation as well. Since, the correction terms minimize the error consistently in comparison with the standard DMD without correction. Further, we can also see that the solution of the DMD-ROM with vanishing viscosity term achieves faster convergence and minimization of the model estimation error.

VI. Conclusions

In this paper, we reported some recent results on stable model reduction for PDEs with parametric uncertainties. We focused on data-driven model reduction, and proposed a DMD-ROM approach. However, one of the main problems with ROMs is their instability, which often occurs due to model truncation and parametric uncertainties. This problem is usually solved using the so called closure models. We formulated the DMD-ROM closure model for uncertain PDEs, in the context of robust nonlinear control, and designed two robust closure models. We then proposed to add an adaptation layer to these DMD-ROM closure models, by using extremum-seeking controllers as real-time tuning algorithms. Finally, we validated our approach on a 1D Burgers equation. The obtained results are interesting, however, we want to investigate in our next steps, the performance of such approach on a more realistic fluid dynamics PDEs, i.e., Navier-Stokes, or Boussinesq equations.

References

[1] S. Vijayshankar, S. Nabi, A. Chakrabarty, P. Grover, and M. Benosman, "Dynamic mode decomposition and robust estimation: Case study of a 2d turbulent boussinesq flow," in *2020 American Control Conference (ACC)*. IEEE, 2020, pp. 2351–2356.

[2] B. Kramer, P. Grover, P. Boufounos, S. Nabi, and M. Benosman, "Sparse sensing and dmd-based identification of flow regimes and bifurcations in complex flows," *SIAM Journal on Applied Dynamical Systems*, vol. 16, no. 2, pp. 1164–1196, 2017.

[3] J. R. Annoni, J. Nichols, and P. J. Seiler, "Wind farm modeling and control using dynamic mode decomposition," in *34th Wind Energy Symposium*, 2016, p. 2201.

[4] K. Kunisch, S. Volkwein, and L. Xie, "Hjb-pod-based feedback design for the optimal control of evolution problems," *SIAM Journal on Applied Dynamical Systems*, vol. 3, no. 4, pp. 701–722, 2004.

[5] I. Mezić, "Spectral properties of dynamical systems, model reduction and decompositions," *Nonlinear Dynamics*, vol. 41, no. 1-3, pp. 309–325, 2005.

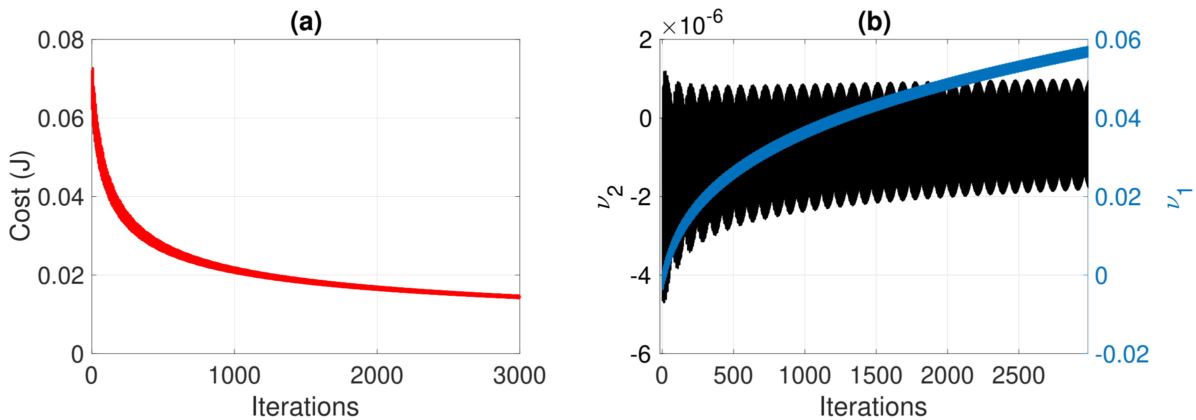

Fig. 1: Extremum seeking cost function in (a), ν_1, ν_2 in (b) as a function of iterations with robust correction.

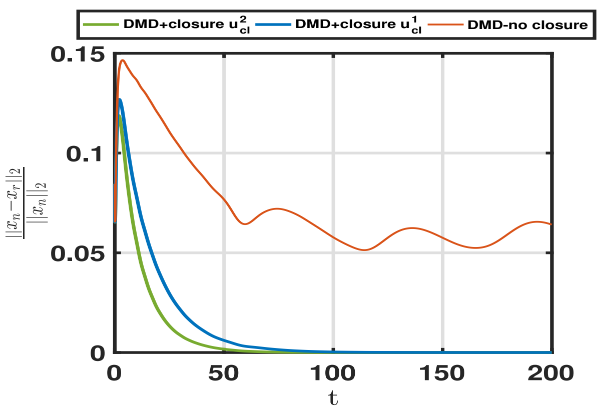

Fig. 2: Reconstruction error for DMD (orange), the robust correction (blue), and the robust correction with vanishing viscosity (green).

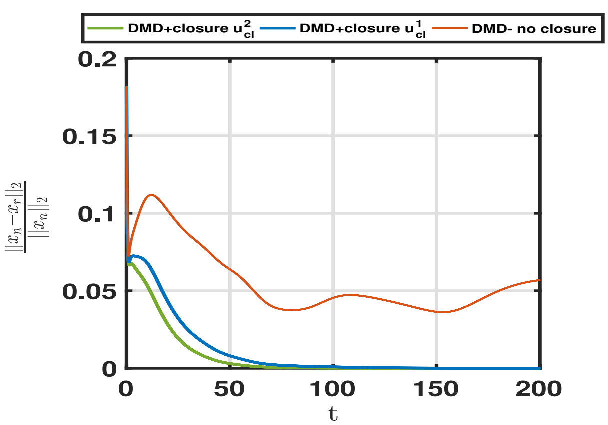

Fig. 3: Same as Fig. 2 but on a truncated system.

[6] B. O. Koopman, “Hamiltonian systems and transformation in hilbert space,” *Proceedings of the national academy of sciences of the united states of america*, vol. 17, no. 5, p. 315, 1931.

[7] L. Cordier, B. R. Noack, G. Tissot, G. Lehnasch, J. Delville, M. Balajewicz, G. Daviller, and R. K. Niven, “Identification strategies for model-based control,” *Experiments in fluids*, vol. 54, no. 8, p. 1580, 2013.

[8] O. San and T. Iliescu, “Proper orthogonal decomposition closure models for fluid flows: Burgers equation,” *International Journal of Numerical Analysis and Modeling*, vol. 1, no. 1, pp. 1–18, 2013.

[9] M. J. Balajewicz, E. H. Dowell, and B. R. Noack, “Low-dimensional modelling of high-reynolds-number shear flows incorporating constraints from the navier-stokes equation,” *Journal of Fluid Mechanics*, vol. 729, p. 285, 2013.

[10] M. Benosman, J. Borggaard, O. San, and B. Kramer, “Learning-based robust stabilization for reduced-order models of 2d and 3d boussinesq equations,” *Applied Mathematical Modelling*, vol. 49, pp. 162–181, 2017.

[11] P. Holmes, J. L. Lumley, and G. Berkooz, *Turbulence, coherent structures, dynamical systems and symmetry*. Cambridge University Press, 1998.

[12] J. N. Kutz, S. L. Brunton, B. W. Brunton, and J. L. Proctor, *Dynamic Mode Decomposition: Data-Driven Modeling of Complex Systems*. Society for Industrial and Applied Mathematics, 2016.

[13] J. H. Tu, C. W. Rowley, D. M. Luchtenburg, S. L. Brunton, and J. N. Kutz, “On dynamic mode decomposition: Theory and applications,” *Journal of Computational Dynamics*, vol. 1, no. 2, pp. 391–421, 2014.

[14] J. Annoni and P. Seiler, “A method to construct reduced-order parameter-varying models,” *International Journal of Robust and Nonlinear Control*, vol. 27, no. 4, pp. 582–597, 2017.

[15] S. Nabi, P. Grover, and C. Caulfield, “Nonlinear optimal control strategies for buoyancy-driven flows in the built environment,” *Computers & Fluids*, vol. 194, p. 104313, 2019.

[16] K. B. Ariyur and M. Krstic, *Real Time Optimization by Extremum Seeking Control*. USA: John Wiley & Sons, Inc., 2003.

[17] M. Benosman, “Extremum Seeking-based Parametric Identification for Partial Differential Equations,” *2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations CPDE*, pp. 19–24, 2016.

[18] ——, *Learning-Based Adaptive Control: An Extremum Seeking Approach - Theory and Applications*. Elsevier Ed., Butterworth-Heinemann, 2016.