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Extremum Seeking-based Parametric Identification for Partial Differential Equations

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Abstract: In this paper we present some results on partial differential equations (PDEs) parametric identification. We follow a deterministic approach and formulate the identification problem as an optimization with respect to unknown parameters of the PDE. We use proper orthogonal decomposition (POD) model reduction theory together with a model free multi-parametric extremum seeking (MES) approach, to solve the identification problem. Finally, the well known Burgers' equation test-bed is used to validate our approach.

1. INTRODUCTION

System identification can be defined as the problem of estimating the best possible model of a system, given a set of experimental data. System identification can be classified as linear vs. nonlinear model identification, time-domain based vs. frequency-domain based, open-loop vs. closed-loop identification, etc. We refer the reader to some outstanding surveys of the field, e.g., Astrom and Eykhoff [1971], Ljung and Vicino [2005], Gevers [2006], Ljung [2010], Pillonetto et al. [2014].

Our focus in this paper is on a specific part of system identification, namely, identification for systems described by PDEs. In this subarea of system identification, we will present some results on a deterministic approach for open-loop parametric identification in the time domain.

PDEs are valuable mathematical models, which are used to describe a large class of systems. For instance, they are used to model fluid dynamics Rowley [2005], Li et al. [2013], MacKunis et al. [2011], Cordier et al. [2013], Balajewicz et al. [2013], or flexible beams and ropes Montseny et al. [1997], Barkana [2014], crowd dynamics Huges [2003], Colombo and Rosini [2005], etc. However, PDEs being infinite dimension systems, are almost impossible to solve in closed-form (except for some exceptions), and are hard to solve numerically, i.e., require a large computation time. Due to this complexity, it is often hard to use PDEs directly to analyze, predict or control systems in real-time. Instead, one viable approach often used in real applications, is to first reduce the PDE model to an ordinary differential equation (ODE) model, which has a finite dimension and then use the obtained ODE to analyze predict or control the system. The step of obtaining an ODE which represents the original PDE as close as possible is known as model reduction, and the obtained ODE is called a reduced order model (ROM). One of the main problems in model reduction is the identification of some unknown parameters of the ROM which also appear in the original PDE, i.e., physical parameters of the system.

Many results have been proposed for PDEs identification. For instance in Xun et al. [2013], the authors proposed two methods to estimate parameters in PDE models: a parameter cascading method and a Bayesian approach. Both methods rely on decomposing the PDE solutions in a linear basis function and then solving an optimization problem in the coefficients of the basis function as well as the PDE parameters to be identified. In Muller and Timmer [2004], two approaches have been investigated, one classified as a regression-based method, where all the terms of the PDE are computed based on measured data, and then the unknown coefficients of the PDE are obtained by solving an algebraic optimization problem, i.e, equaling both sides of the PDE equation. The second method can be classified as a dynamical approaches, in the sense that the unknown parameters of the PDE are obtained by solving an optimization problem which minimizes the distance between the measured data and the solutions of the PDE over time. Many other work on PDE identification fall into one of these two categories, e.g., refer to Parlitz and C.Merkwirth [2000], Voss et al. [1999] for some regression-based identification techniques, and Baake et al. [1992], Muller and Timmer [2002] for a dynamical approach for PDEs identification.

In this paper, we propose an alternative method, which might be classified as a dynamical approach. Indeed, we follow here the deterministic identification formulation of Ljung and Glad [1994], in the sense that we deal with nonlinear infinite dimensional models in the deterministic time domain. We use POD model reduction theory together with a model-free optimization approach to solve the identification problem. We formulate the identification problem as a minimization of a performance cost function, and use the extremum seeking theory to solve the optimization problem online, leading to a simple real-time solution for open-loop parametric identification for PDEs.

This paper is organized as follows: we first introduce some notations and definitions in Section 2. Section 3 is dedicated to the problem formulation and the presentation of the proposed solution. The case of the Burgers' equation is studied in Section 4. Finally, a conclusion is presented in Section 5.

2. BASIC NOTATIONS AND DEFINITIONS

Throughout the paper we will use $\mathbb{N}$ to denote the set of natural numbers, $\|.\|$ to denote the Euclidean vector norm; i.e., for $x \in \mathbb{R}^n$ we have $\|x\| = \sqrt{x^T x}$. The Kronecker delta function is defined as: $\delta_{ij} = 0,$ for $i \neq j$ and $\delta_{ii} = 1$. We will use $\dot{f}$ for the short notation of time derivative of f, and x^T for the transpose of a vector x. A function is said analytic in a given set, if it admits a convergent Taylor series approximation in some neighborhood of every point of the set. We consider the Hilbert space $\mathcal{Z} = L^2([0,1])$, which is the space of Lebesgue square integrable functions, i.e., $f \in \mathcal{Z}$, iff $\int_0^1 |f(x)|^2 dx < \infty$. We define on $\mathcal{Z}$ the inner product $\langle \cdot, \cdot \rangle_{\mathcal{Z}}$ and the associated norm $\|.\|_{\mathcal{Z}}$, as $\langle f, g \rangle_{\mathcal{Z}} = \int_0^1 f(x)g(x)dx$, for $f, g \in \mathcal{Z}$, and $\|f\|^2_{\mathcal{Z}} = \int_0^1 |f(x)|^2 dx$. A function $\omega(t,x)$ is in $L^2([0,T];\mathcal{Z})$ if for each $0 \leq t \leq T$, $\omega(t,\cdot) \in \mathcal{Z}$, and $\int_0^T \|\omega(t,\cdot)\|^2_{\mathcal{Z}} dt \leq \infty$.

Definition 1. (Haddad and Chellaboina [2008]). A system $\dot{x} = f(t,x)$ is said to be *Lagrange stable* if for every initial condition x_0 associated with the time instant t_0, there exists $\epsilon(x_0)$, such that $\|x(t)\| < \epsilon$, $\forall t \geq t_0 \geq 0$.

3. IDENTIFICATION OF PDE MODELS BY EXTREMUM SEEKING

3.1 MES-based ROM parameters identification

Consider a stable dynamical system modelled by a nonlinear PDE of the form

$$\dot{z} = \mathcal{F}(z,p) \in \mathcal{Z}, \tag{1}$$

where $\mathcal{Z}$ is an infinite-dimension Hilbert space, and $p \in \mathbb{R}^m$ represents the vector of physical parameters to be identified. While solutions to this PDE can be obtained through numerical discretization, e.g., finite elements, finite volumes, finite differences, etc., these computations are often very expensive and not suitable for online applications, e.g., airflow analysis, prediction and control. However, solutions of the original PDE often exhibit low rank representations in an 'optimal' basis, which is exploited to reduce the PDE to a finite dimension ODE.

The general idea is as follows: one first finds a set of 'optimal' (spatial) basis vectors $\phi_i \in \mathbb{R}^n$ (the dimension n is generally very large and comes form a 'brut-force' discretization of the PDE, e.g., finite element discretization), and then approximates the PDE solution as

$$z(t) \approx \Phi z_r(t) = \sum_{i=1}^{r} q_i(t)\phi_i, \tag{2}$$

where Φ is a $n \times r$ matrix containing the basis vectors ϕ_i as column vectors. Next, the PDE equation is projected into the finite r-dimensional space via classical nonlinear model reduction techniques, e.g., Galerkin projection, to obtain a ROM of the form

$$\dot{q}(t) = F(q(t),p) \in \mathbb{R}^r, \tag{3}$$

where $F : \mathbb{R}^r \rightarrow \mathbb{R}^r$ is obtained from the original PDE structure, through the model reduction technique, e.g., the Galerkin projection. Clearly, the problem lies in the selection of this 'optimal' basis matrix Φ. There are many model reduction methods to find the projection basis functions for nonlinear systems. For example proper orthogonal decomposition (POD), dynamic mode decomposition (DMD), and reduced basis (RB) are some of the most used methods. We will recall hereafter the POD method, however, we believe that the MES-based identification results are independent of the type of model reduction approach, and the results of this paper remain valid regardless of the selected model reduction method.

POD model reduction: We give here a brief recall of POD basis functions computation, the interested reader can refer to Kunisch and Volkwein [2007], Gunzburger et al. [2007] for a more complete presentation about POD theory.

We consider here the case where the POD basis are computed mathematically from approximation solutions snapshot of the PDE. The general idea behind POD is to select a set of basis functions that capture an optimal amount of energy of the original PDE. The POD basis are obtained from a collection of snapshots over a finite time support of the PDE solutions. In the context of this work, these snapshots are obtained by solving an approximation (discretization) of the PDE equation, e.g., using finite element method (FEM). The POD basis functions computation steps are presented below in more details.

First, the original PDE is discretized using any finite element basis functions, e.g., piecewise linear functions or spline functions, etc. (we are not presenting here any FEM method, instead we refer the reader to the numerous manuscripts in the field of FEM, e.g., Sordalen [1997], Fletcher [1983]). Let us denote the associated PDE solutions approximation by $z_{fem}(t,x)$, where t stands for the scalar time variable, and x stands for the space variable. We consider here (for simplicity of the notations) the case of one dimension where x is a scalar in a finite interval, which we consider, without loss of generality, to be [0, 1]. Next, we compute a set of s snapshots of approximates solutions as

$$S_z = \{z_{fem}(t_1,.), ..., z_{fem}(t_s,.)\} \subset \mathbb{R}^n, \tag{4}$$

where n is the selected number of FEM basis functions. Now we define the so called correlation matrix K^z elements as

$$K^z{}_{ij} = \frac{1}{s} < z_{fem}(t_i,.), z_{fem}(t_j,.) >, \; i,j = 1,...,s. \tag{5}$$

We then compute the normalized eigenvalues and eigenvectors of K^z, denoted as λ^z, and v^z. Note that λ^z are also referred to as the POD eigenvalues. Eventually, the i^{th} POD basis function is given by

$$\phi_i^{pod}(x) = \frac{1}{\sqrt{s}\sqrt{\lambda_i^z}} \sum_{j=1}^{j=s} v_i^z(j) z^{fem}(t_j,x), \; i = 1,...,n_{pod}, \tag{6}$$

where $n_{pod} \leq s$ is the number of retained POD basis functions, which depends on the application.

One of the main properties of the POD basis functions is orthonormality, which means that the basis satisfy the following equalities

$$< \phi_i^{pod}, \phi_j^{pod} > = \int_0^1 \phi_i^{pod}(x)\phi_j^{pod}(x)dx = \delta_{ij}, \tag{7}$$

where δ_{ij} denotes the Kronecker delta function. The solution of the PDE (1) can then be approximated as

$$z^{pod}(t,x) = \sum_{i=1}^{i=n_{pod}} \phi_i^{pod}(x) q_i^{pod}(t), \tag{8}$$

where q_i^{pod}, $i = 1,...,n_{pod}$ are the POD projection coefficients (which play the role of the z^r in the ROM (3)). Finally, the PDE (1) is projected on the reduced dimension POD space using a Galerkin projection, i.e., both sides of equation (1) are multiplied by the POD basis functions, where z is substituted by z^{pod}, and then both sides are integrated over the space interval [0, 1], which using the orthonormality constraints (7) and the boundary constraints of the original PDE, leads to an ODE of the form

$$\dot{q}^{pod}(t) = F(q^{pod}(t),p) \in \mathbb{R}^{n_{pod}}, \tag{9}$$

where the structure (in terms of nonlinearities) of the vector field F is related to the structure of the original PDE, and where $p \in \mathbb{R}^m$ represents the vector of parametric uncertainties to be identified.

We can now proceed with the MES-based identification of the parametric uncertainties.

MES-based PDEs open-loop parameters estimation: We will use here an MES algorithm to estimate the PDE's parametric uncertainties, using its reduced order model; the POD ROM. First, we need to introduce some basic stability assumptions.

Assumption 1. The solutions of the original PDE model (1) are assumed to be in $L^2([0,\infty);\mathcal{Z})$, and the associated POD reduced order model (8), (9) is Lagrange stable.

Remark 1. Assumption 1 is needed to be able to perform open-loop identification of the system, without the need for any feedback stabilization.

Now, to be able to use the MES framework to identify the parameters vector p, we define an identification cost function as

$$Q(\hat{p}) = H(e_z(\hat{p})), \tag{10}$$

where $\hat{p}$ denotes the estimate of p, H is a positive definite function of e_z, and e_z represents the error between the ROM model (8),(9) and the system's measurements z_m, defined as

$$e_z(t) = z^{pod}(t,x_m) - z_m(t,x_m,\varsigma), \tag{11}$$

x_m being the points in space where the measurements are obtained, and ς represents additive white measurement noise.

To derive an upper bound on the estimation error norm, we add the following assumptions of the cost function Q and its variation with respect to the parameters $\hat{p}$.

Assumption 2. The cost function Q has a local minimum at $\hat{p}^* = p$.

Assumption 3. The original parameters estimates vector $\hat{p}$, i.e., the nominal parameters value, is close enough to the actual parameters vector p.

Assumption 4. The cost function is analytic and its variation with respect to the uncertain variables is bounded in the neighborhood of p^*, i.e., $\|\frac{\partial Q}{\partial p}(\tilde{p})\| \leq \xi_2,\ \xi_2 > 0,\ \tilde{p} \in \mathcal{V}(p^*)$, where $\mathcal{V}(p^*)$ denotes a compact neighborhood of p^*.

Based on the above assumptions, we can summarize the open-loop identification result in the following Lemma.

Lemma 1. Consider the system (1), then under Assumptions 2, 3, and 4, the uncertain parameters vector p can be estimated online using the algorithm

$$\hat{p}(t) = p_{nom} + \Delta p(t), \tag{12}$$

where p_{nom} is the nominal value of p, $\Delta p = [\delta p_1, ..., \delta p_m]^T$ is computed using the MES algorithm

$$\begin{aligned} \dot{y}_i &= a_i sin(\omega_i t + \frac{\pi}{2}) Q(\hat{p}), \\ \delta p_i &= y_i + a_i sin(\omega_i t - \frac{\pi}{2}),\ i \in \{1,...,m\} \end{aligned} \tag{13}$$

with $\omega_i \neq \omega_j,\ \omega_i + \omega_j \neq \omega_k,\ i,j,k \in \{1,...,m\}$, and $\omega_i > \omega^*,\ \forall i \in \{1,...,m\}$, with ω^* large enough, and Q given by (10), (11), with the estimate upper-bound

$$\|e_p(t)\| = \|\hat{p} - p\| \leq \frac{\xi_1}{\omega_0} + \sqrt{\sum_{i=1}^{i=m} a_i^2},\ t \to \infty, \tag{14}$$

where $\xi_1 > 0$, and $\omega_0 = \max_{i\in\{1,...,m\}} \omega_i$.

Proof 1. First, based on Assumptions 2, 3 and 4, the extremum seeking nonlinear dynamics (13), can be approximated by a linear averaged dynamic (using averaging approximation over time, [Rotea, 2000, p. 435, Definition 1]). Furthermore, $\exists \xi_1,\ \omega^*$, such that for all $\omega_0 = \max_{i\in\{1,...,m\}} \omega_i > \omega^*$, the solution of the averaged model $\Delta p_{aver}(t)$ is locally close to the solution of the original MES dynamics, and satisfies [Rotea, 2000, p. 436]

$$\|\Delta p(t) - d(t) - \Delta p_{aver}(t)\| \leq \frac{\xi_1}{\omega_0},\ \xi_1 > 0,\ \forall t \geq 0,$$

with $d(t) = (a_1 sin(\omega_1 t - \frac{\pi}{2}), ..., a_m sin(\omega_m t - \frac{\pi}{2}))^T$. Moreover, since Q is analytic it can be approximated locally in $\mathcal{V}(\beta^*)$ with a quadratic function, e.g., Taylor series up to second order, which leads to [Rotea, 2000, p. 437]

$$lim_{t\to\infty} \Delta p_{aver}(t) = \Delta p^*,$$

such that

$$\Delta p^* + p_{nom} = p,$$

which together with the previous inequality leads to

$$\|\Delta p(t) - \Delta p^*\| - \|d(t)\| \leq |\Delta p(t) - \Delta p^* - d(t)\| \leq \frac{\xi_1}{\omega_0}, \quad \xi_1 > 0,\ t \to \infty,$$

$$\Rightarrow \|\Delta p(t) - \Delta p^*\| \leq \frac{\xi_1}{\omega_0} + \|d(t)\|,\ t \to \infty.$$

This finally implies that

$$\|\Delta p(t) - \Delta p^*\| \leq \frac{\xi_1}{\omega_0} + \sqrt{\sum_{i=1}^{i=m} a_i^2},\ \xi_1 > 0,\ t \to \infty.$$

Remark 2. One of the main advantages of using MES to solve the identification optimal problem, is that it is a model-free optimization algorithm which needs only one measurement at a time to direct the search of the optimal parameter. Furthermore, dither-based MES is well known to be robust to measurement noise, e.g., Calli et al. [2012], which makes it a good candidate for solving identification problems, where measurements are often contaminated with noise, e.g., Ljung and Vicino [2005].

4. THE COUPLED BURGERS PDE EQUATION

We consider here the case of the coupled Burgers' equation, e.g., Kramer [2011]

$$\begin{cases} \dfrac{\partial \omega(t,x)}{\partial t} + \omega(t,x)\dfrac{\omega(t,x)}{\partial x} = \mu \dfrac{\partial^2 \omega(t,x)}{\partial x^2} - \kappa T(t,x), \\ \dfrac{\partial T(t,x)}{\partial t} + \omega(t,x)\dfrac{\partial T(t,x)}{\partial x} = c\dfrac{\partial^2 T(t,x)}{\partial x^2} + f(t,x), \end{cases} \tag{15}$$

where T represents the temperature and ω represents the velocity field, κ is the coefficient of the thermal expansion, c the heat diffusion coefficient, μ the viscosity coefficient (inverse of the Reynolds number R_e), x is the one dimensional space variable $x \in [0,1]$, $t > 0$, and f is the external forcing term such that $f \in L^2((0,\infty), X),\ X = L^2([0,1])$. The previous equation is associated with the following boundary conditions

$$\begin{aligned} \omega(t,0) = \delta_1,\ \frac{\partial \omega(t,1)}{\partial x} = \delta_2, \\ T(t,0) = T_1,\ T(t,1) = T_2, \end{aligned} \tag{16}$$

where $\delta_1,\ \delta_2,\ T_1,\ T_2 \in \mathbb{R}_{\geq 0}$.

We consider here the following general initial conditions

$$\begin{aligned} \omega(0,x) = \omega_0(x) \in L^2([0,1]), \\ T(0,x) = T_0(x) \in L^2([0,1]). \end{aligned} \tag{17}$$

Following a Galerkin-type projection into POD basis functions, e.g., Kramer [2011], the coupled Burgers' equation is reduced to a POD ROM with the following structure

$$\begin{pmatrix} \dot{q}^{pod}_{\omega} \\ \dot{q}^{pod}_{T} \end{pmatrix} = B_1 + \mu B_2 + \mu\ D\ q^{pod} + \tilde{D} q^{pod} + [C q^{pod}] q^{pod},$$
$$\omega_{ROM}(x,t) = \omega_0(x) + \sum_{i=1}^{i=N_{pod\omega}} \phi(x)^{pod}_{\omega i} q^{pod}_{\omega i}(t),$$
$$T_{ROM}(x,t) = T_0(x) + \sum_{i=1}^{i=N_{podT}} \phi(x)^{pod}_{Ti} q^{pod}_{Ti}(t), \quad (18)$$

where matrix B_1 is due to the projection of the forcing term f, matrix B_2 is due to the projection of the boundary conditions, matrix D is due to the projection of the viscosity damping term $\mu \frac{\partial^2 \omega(t,x)}{\partial x^2}$, matrix $\tilde{D}$ is due to the projection of the thermal coupling and the heat diffusion terms $-\kappa T(t,x)$, $c\frac{\partial^2 T(t,x)}{\partial x^2}$, and the matrix C is a three-dimensional tensor due to the projection of the gradient-based terms $\omega \frac{\omega(t,x)}{\partial x}$, and $\omega \frac{\partial T(t,x)}{\partial x}$. The notations $\phi^{pod}_{\omega i}(x)$, $q^{pod}_{\omega i}(t)$ $(i = 1, ..., N_{pod\omega})$, $\phi^{pod}_{Ti}(x)$, $q^{pod}_{Ti}(t)$ $(i = 1, ..., n_{podT})$, stand for the space basis functions and the time projection coordinates, for the velocity and the temperature, respectively. $\omega_0(x)$, $T_0(x)$ represent the mean values (over time) of ω and T, respectively.

To illustrate the MES-based PDEs parameters estimation results presented in Section 3, we consider here the case of the Burgers' equation with an uncertainty on the Reynolds number R_e (other cases with uncertainties on c, and κ have testes as well, but could not be reported here due to number of pages limitations, we will however report them in a longer journal version of this work). We consider the coupled Burgers' equation (15), with the parameters $R_e = 1000$, $\kappa = -1$, $c = 0.01$, the boundary conditions $\delta_1 = 0$, $\delta_2 = 5$, $T_1 = 0$, $T_2 = 0.1 sin(0.5\pi t)$, the initial conditions $\omega_0(x) = 2(x^2(0.5 - x)^2)$, $T_0(x) = 0.5 sin(\pi x)^5$, and a zero forcing term f. We assume a large uncertainty on R_e, and consider that its known value is $R_{e-nom} = 50$. We apply the discrete version of the estimation algorithm of Lemma 1. We estimate the value of R_e, as follows

$$\hat{R}_e(t) = R_{e-nom} + \delta R_e(t),$$
$$\delta R_e(t) = \delta \hat{R}_e((I-1)t_f),\ (I-1)t_f \le t < I t_f,\ I \in \mathbb{N}, \quad (19)$$

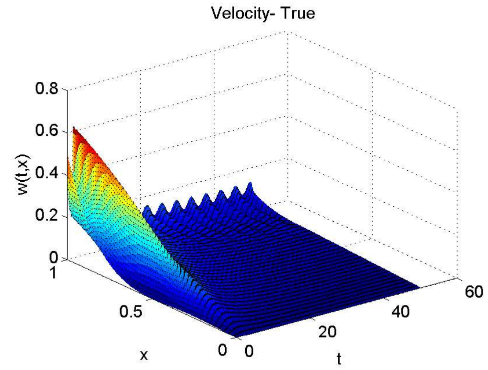
where I is the learning iteration number, $t_f = 50\ sec$ the time horizon of one learning iteration, and $\delta \hat{R}_e$ is computed using the iterative MES algorithm

$$\dot{y} = a sin(\omega t + \frac{\pi}{2}) Q(\hat{R}_e),$$
$$\delta \hat{R}_e = y_i + a sin(\omega t - \frac{\pi}{2}). \quad (20)$$

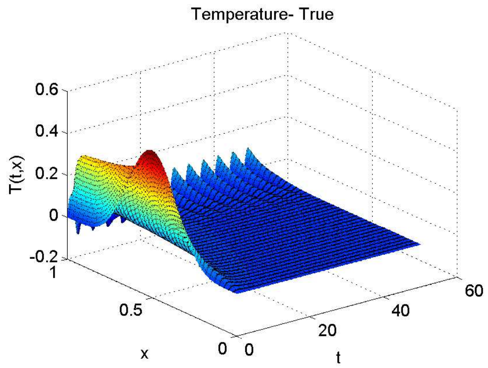
We choose the learning cost function as

$$Q = Q_1 \int_0^{t_f} < e_T, e_T > dt + Q_2 \int_0^{t_f} < e_\omega, e_\omega > dt, \quad (21)$$

with Q_1, $Q_2 > 0$, $e_T = T - T_{ROM}$, $e_\omega = \omega - \omega_{ROM}$ define the errors between the measurements and the POD ROM solution for temperature and velocity, respectively. We assume that the measurements are corrupted with additive white noise with standard deviation $\sigma = 10^{-2}$. We applied the ES algorithm (20), (21), with $a = 0.0178$, $\omega = 10\ rad/sec$, $Q_1 = Q_2 = 1$. For the evaluation of the cost function in (21), in this paper, we simulate the case of limited number of sensors, where we assume that we only have 10 measurements for the velocity and 10 measurements for the temperature, uniformly distributed over $[0, 1]$.



(a) True velocity profile

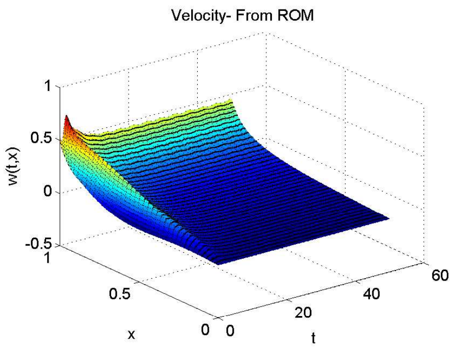


(b) True temperature profile

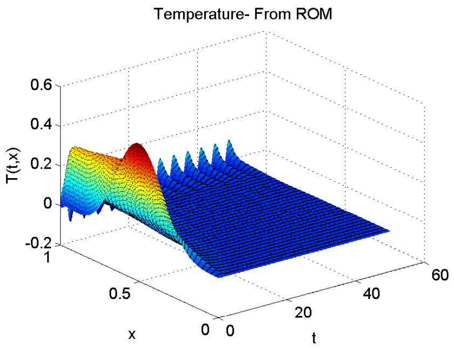
Fig. 1. True solutions of (15)

We first show in Figure 1, the plots of the true (obtained by solving the Burgers' PDE with finite elements method, with a uniform grid of 100 elements in time and space [1]). Next, we show in Figure 2, the velocity and temperature profiles, obtained by the nominal, i.e., learning-free POD ROM with 4 POD modes for the velocity and 4 modes for the temperature, considering the incorrect value $R_e = 50$. From Figures 1, 2, we can see that the temperature profile obtained by the nominal POD ROM is not too different from the true profile. However, the velocity profiles are different, which is due to the fact that the uncertainty of R_e affects mainly the velocity part of the PDE. The error between the true solutions and the nominal POD ROM solutions are displayed in Figure 3. Now, we show the MES-based learning of the uncertain parameter R_e. We first report in Figure 4(a), the learning cost function over the learning iterations. We notice that, with the chosen learning parameters a, ω, the MES exhibits a big exploration step after the first iteration, which leads to a large cost function first. However, this large value of the cost function (due to the large exploration step), leads quickly to the neighborhood of the true value of R_e, as seen in Figure 4(b). The error between the POD ROM after learning and the true solutions are depicted in Figure 5. By comparing Figure 3 and Figure 5, we can see that the error between the POD ROM solutions and the true solutions have been largely reduced with the learning of the actual value of R_e, i.e., when $\hat{R}_e$ converges to a small neighborhood of the true value of R_e.

[1] We thank here Dr. Boris Kramer, former intern at MERL, for sharing his codes to solve the Burgers' equation.

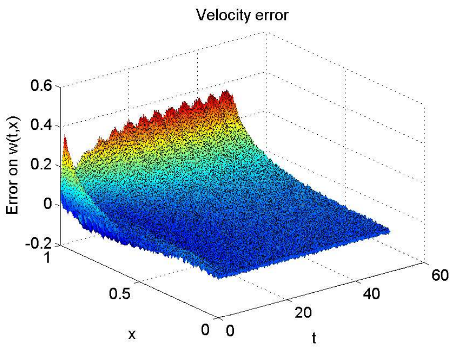


(a) Learning-free POD ROM velocity profile

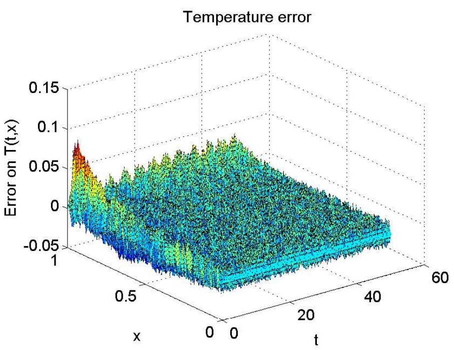


(b) Learning-free POD ROM temperature profile

Fig. 2. Learning-free POD ROM solutions of (15)

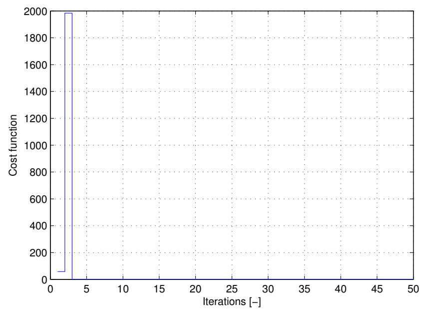


(a) Error between the true velocity and the learning-free POD ROM velocity profile

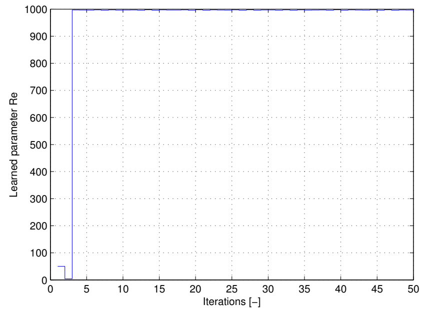


(b) Error between the true temperature and the learning-free POD ROM temperature profile

Fig. 3. Errors between the nominal POD ROM and the true solutions

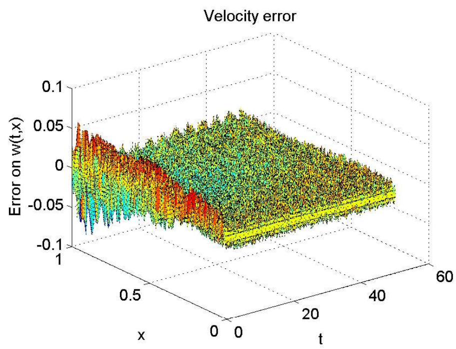


(a) Learning cost function vs. number of iterations

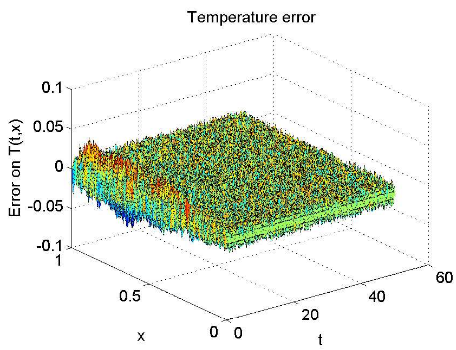


(b) Learned parameter $\hat{R_e}$ vs. number of iterations

Fig. 4. Learned parameters and learning cost function



(a) Error between the true velocity and the learning-based POD ROM velocity profile



(b) Error between the true temperature and the learning-based POD ROM temperature profile

Fig. 5. Errors between the learning-based POD ROM and the true solutions

5. CONCLUSION

In this work we have studied the problem of PDEs parametric identification. We have formulated the problem as an optimization with respect the unknown parameters, and have proposed to use model-free extremum seeking theory to search for the optimal PDE parameters. We believe that one of the main advantages of using extremum seeking theory for parametric identification is the fact that extremum seeking requires only one measurement at any given time to direct the search for the optimal parameters. Furthermore, the proposed extremum seeking algorithm, namely, the dither-based algorithm is well know to be robust with respect to measurement noises, which makes it a good candidate for solving identification problems, where measurements are often contaminated with noise. In this context, we have proposed to merge together POD model reduction theory and extremum seeking theory to propose a solution for PDEs parametric identification. Even though, these preliminary results are satisfactory, we believe that this direction can be developed by looking at convexification methods, i.e., change of coordinates in the unknown parameters. Another improvement direction, could be to use other type of model-free optimization algorithms, like semi-global extremum seeking algorithm, reinforcement learning algorithms, etc. We are investigating some of these directions and will communicate the obtained results in our future reports.

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