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TR2016-043 July 2016

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2016 American Control Conference (ACC)

Indirect Adaptive Model Predictive Control for Linear Systems with Polytopic Uncertainty

Stefano Di Cairano

***Abstract*— We develop an indirect adaptive model predictive control algorithm for uncertain linear systems subject to constraints. The system is modeled as a polytopic linear parameter varying system where the convex combination vector is constant but unknown. The terminal cost and set are constructed from a parameter-dependent Lyapunov function and the associated control law, and robust control invariant set constraints are enforced. The proposed design ensures robust constraint satisfaction and recursive feasibility, is input-to-state stable with respect to the parameter estimation error and it only requires the online solution of quadratic programs.**

I. INTRODUCTION

Often in Model Predictive Control (MPC) [1], some of the model parameters are uncertain at design time, especially in factory automation, automotive, and aerospace applications [2], [3], where the control algorithm needs also to have low complexity and computational effort. For uncertain models, robust MPC methods have been proposed, see, e.g., [4]–[7]. Some of the limitations of these methods are either in the computational cost, due to solving linear matrix inequalities (LMIs) at each control step [4]–[6], or in applying only to additive disturbances [7]. These limitations are often due to considering constantly changing parameters.

Alternatively, when the parameters are unknown but constant or slowly varying, one can learn their values. Adaptive MPC algorithms have been recently proposed based on different methods, such as min-max approaches [8], learning of constant offsets [9], and set membership identification [10]. Another class of adaptive MPC algorithms is related to controlling the uncertain system while guaranteeing sufficient excitation for identification, see, e.g., [11], [12].

In this paper we propose a MPC design that operates concurrently with a parameter estimation scheme, thus resulting in an indirect adaptive MPC (IAMPC, for shortness) approach, that retains constraint satisfaction guarantees and certain stability properties. Motivated by the case of the unknown but constant (or slowly varying) parameters and by the need to limit the computational burden, here we do not seek robust stability, but robust constraint satisfaction and an input-to-state stable (ISS) closed-loop with respect to the estimation error. ISS will hold with only minimal assumptions on the estimates, and if the correct parameter value will be eventually estimated the closed-loop will become asymptotically stable (AS). Constraint satisfaction is guaranteed even if the parameters change continuously.

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We build upon the results in [13] for calibrating the MPC model after deployment, yet not concurrently to the plant operation. For uncertain systems represented as polytopic linear difference inclusions (pLDI) we design a parameter-dependent quadratic terminal cost and a robust terminal constraint using a parameter-dependent Lyapunov function (pLF) [14] and its corresponding stabilizing control law. Robust constraint satisfaction in presence of parameter estimation error is obtained by enforcing robust control invariant set constraints [15]. A parameter prediction update law is also designed to ensure the desired properties. The IAMPC allows uncertainty in the system dynamics, as opposed to additive disturbances in [7], [9], and only solves quadratic programs (QPs), as opposed to LMIs in [4]–[6].

The paper is structured as follows. After the preliminaries in Section II, in Section III we design the cost function ensuring the unconstrained IAMPC to be ISS with respect to the estimation error. For constrained IAMPC, in Section IV we design the terminal set and the robust constraints ensuring recursive feasibility. In Section V we combine the cost function and the constraints with a parameter estimate prediction update. In Section VI we show a numerical example and a case study in air conditioning control. Conclusions and future developments are discussed in Section VII.

Notation: $\mathbb{R}$, $\mathbb{R}_{0+}$, $\mathbb{R}_{+}$, $\mathbb{Z}$, $\mathbb{Z}_{0+}$, $\mathbb{Z}_{+}$ are the sets of real, nonnegative real, positive real, and integer, nonnegative integer, positive integer numbers. We denote interval of numbers using notations like $\mathbb{Z}_{[a,b)} = \{z \in \mathbb{Z} : a \leq z < b\}$. $\mathrm{co}\{\mathcal{X}\}$, and $\mathrm{int}\{\mathcal{X}\}$ denote the convex hull and the interior of set $\mathcal{X}$. For vectors, inequalities are intended componentwise, while for matrices indicate (semi)definiteness, and $\lambda_{\min}(Q)$ is the smallest eigenvalue of Q. By $[x]_i$ we denote the i-th component of vector x, and by I and 0 the identity and the "all-zero" matrices of appropriate dimension. $\|\cdot\|_p$ denotes the p-norm, and $\|\cdot\| = \|\cdot\|_2$. For a discrete-time signal $x \in \mathbb{R}^n$ with sampling period T_s, $x(t)$ is the value a sampling instant t, i.e., at time $T_s t$, $x_{k|t}$ denotes the predicted value at sample $t+k$, i.e., $x(t+k)$, based on data at sample t, and $x_{0|t} = x(t)$. A function $\alpha : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, $\alpha(0) = 0$; if in addition $\lim_{c\to\infty} \alpha(c) = \infty$, α is of class $\mathcal{K}_\infty$.

II. PRELIMINARIES AND PROBLEM DEFINITION

We review some standard definitions and results. For details, see, e.g., [1, Appendix B].

Definition 1: Given $x(t+1) = f(x(t), w(t))$, $x \in \mathbb{R}^n$, $w \in \mathcal{W} \subseteq \mathbb{R}^d$, a set $\mathcal{S} \subset \mathbb{R}^n$ is robust positive invariant

(RPI) for f iff for all $x \in \mathcal{S}$, $f(x,w) \in \mathcal{S}$, for all $w \in \mathcal{W}$. If $w = \{0\}$, $\mathcal{S}$ is called positive invariant (PI). □

Definition 2: Given $x(t+1) = f(x(t), u(t), w(t))$, $x \in \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, $w \in \mathcal{W} \subseteq \mathbb{R}^d$, a set $\mathcal{S} \subset \mathbb{R}^n$ is robust control invariant (RCI) for f iff for all $x \in \mathcal{S}$, there exists $u \in \mathcal{U}$ such that $f(x,u,w) \in \mathcal{S}$, for all $w \in \mathcal{W}$. If $w = \{0\}$, $\mathcal{S}$ is called control invariant (CI). □

Definition 3: Given $x(t+1) = f(x(t))$, $x \in \mathbb{R}^n$, and a PI set $\mathcal{S}$ for f, $0 \in \mathcal{S}$, a function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}_{0+}$ such that there exists $\alpha_1, \alpha_2, \alpha_\Delta \in \mathcal{K}_\infty$ such that $\alpha_1(\|x\|) \leq \mathcal{V}(x) \leq \alpha_2(\|x\|)$, $\mathcal{V}(f(x)) - \mathcal{V}(x) \leq -\alpha_\Delta(\|x\|)$ for all $x \in \mathcal{S}$ is a Lyapunov function for f in $\mathcal{S}$. □

Definition 4: Given $x(t+1) = f(x(t), w(t))$, $x \in \mathbb{R}^n$, $w \in \mathcal{W} \subseteq \mathbb{R}^d$, and a RPI set $\mathcal{S}$ for f, $0 \in \mathcal{S}$, a function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}_+$ such that there exists $\alpha_1, \alpha_2, \alpha_\Delta \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$ such that $\alpha_1(\|x\|) \leq \mathcal{V}(x) \leq \alpha_2(\|x\|)$, $\mathcal{V}(f(x)) - \mathcal{V}(x) \leq -\alpha_\Delta(\|x\|) + \gamma(\|w\|)$ for all $x \in \mathcal{S}$, $w \in \mathcal{W}$ is an input-to-state stable (ISS) Lyapunov function for f in $\mathcal{S}$ with respect to w.

Result 1: Given $x(t+1) = f(x(t))$, $x \in \mathbb{R}^n$, and a PI set $\mathcal{S}$ for f, $0 \in \mathcal{S}$, if there exists a Lyapunov function for f in $\mathcal{S}$, the origin is asymptotically stable (AS) for f with domain of attraction $\mathcal{S}$. Given $x(t+1) = f(x(t), w(t))$, $x \in \mathbb{R}^n$, $w \in \mathcal{W} \subseteq \mathbb{R}^d$, and a RPI set $\mathcal{S}$ for f, $0 \in \mathcal{S}$, if there exists a ISS Lyapunov function for f in $\mathcal{S}$, the origin is ISS for f with respect to w with domain of attraction $\mathcal{S}$.

We consider the uncertain constrained discrete-time system with sampling period T_s,

$$x(t+1) = \sum_{i=1}^{\ell} [\bar{\xi}]_i A_i x(t) + Bu(t), \tag{1a}$$

$$x \in \mathcal{X}, \ u \in \mathcal{U} \tag{1b}$$

where $A_i \in \mathbb{R}^{n \times n}$, $i \in \mathbb{Z}_{[1,\ell]}$ and B are known matrices of appropriate size, and $\mathcal{X} \subseteq \mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^m$ are constraints on system states and inputs. In (1), the uncertainty is associated to $\bar{\xi} \in \Xi \subset \mathbb{R}^\ell$, which is unknown and constant or changing slowly with respect to the system dynamics, and $\Xi = \{\xi \in \mathbb{R}^\ell : 0 \leq [\xi]_i \leq 1, \sum_{i=1}^{\ell} [\xi]_i = 1\}$. We call $\bar{\xi}$ *convex combination vector*, since it describes a convex combination of the "vertex systems" $f_i(x,u) = A_i x + Bu$, $i \in \mathbb{Z}_{[1,\ell]}$.

Assumption 1: An estimator is computing the estimate $\xi(t)$ of $\bar{\xi}$ such that $\xi(t) \in \Xi$ for all $t \in \mathbb{Z}_{0+}$.

We denote by $\tilde{\xi}(t) = \bar{\xi}(t) - \xi$, the estimation error at time t for which $\tilde{\xi}(t) + \xi \in \Xi$. Given $\xi \in \Xi$, for shortness we write $\tilde{\xi} \in \tilde{\Xi}(\xi)$, where $\tilde{\Xi}(\xi) = \{\tilde{\xi} \in \mathbb{R}^\ell : \exists \bar{\xi} \in \Xi, \ \tilde{\xi} = \bar{\xi} - \xi\}$ is the set of possible estimation error vectors. Assumption 1 is the only estimator property required for the developments in this paper to hold. Some comments on how to design estimators that satisfy Assumption 1 are given later, in Section V.

Remark 1: The trajectories produced by (1a) are a subset of those of the pLDI

$$x(t+1) \in \mathrm{co}\{A_i x(t) + Bu(t)\}_{i=1}^{\ell}. \tag{2}$$

The pLDI (2) is equivalent to (1a) if a varying parameter vector, i.e., $\bar{\xi}(t) \in \Xi$, is considered.

Consider the finite time optimal control problem

$$\mathcal{V}^{\mathrm{MPC}}_{\xi_t^N}(x(t)) = \min_{U_t} \quad x'_{N|t} \mathcal{P}(\xi_{N|t}) x_{N|t} + \tag{3a}$$

$$\sum_{k=0}^{N-1} x'_{k|t} Q x_{k|t} + u'_{k|t} R u_{k|t} \tag{3b}$$

$$\text{s.t. } x_{k+1|t} = \sum_{i=1}^{\ell} [\xi_{k|t}]_i A_i x_{k|t} + B u_{k|t} \tag{3c}$$

$$u_{k|t} \in \mathcal{U}, \ x_{k|t} \in \mathcal{X} \tag{3d}$$

$$(x_{k|t}, u_{k|t}) \in \mathcal{C}_{xu} \tag{3e}$$

$$x_{N|t} \in \mathcal{X}_N \tag{3f}$$

$$x_{0|t} = x(t), \tag{3g}$$

where $N \in \mathbb{R}_+$ is the prediction horizon, $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$, $Q, R > 0$, $\mathcal{P}(\xi) \in \mathbb{R}^{n \times n}$, $\mathcal{P}(\xi) > 0$, for all $\xi \in \Xi$, $\mathcal{C}_{x,u} \subseteq \mathcal{X} \times \mathcal{U}$, $U_t = [u_{0|t} \ \ldots u_{N-1|t}]$ is the sequence of control inputs along the prediction horizon, and $\xi_t^N = [\xi_{0|t} \ \ldots \xi_{N|t}] \in \Xi^{N+1}$ is a sequence of predicted parameters, not necessarily constant. Let $U_t^* = [u^*_{0|t} \ \ldots u^*_{N-1|t}]$ be the solution of (3) at $t \in \mathbb{Z}_{0+}$.

Problem 1: Given (1) and an estimator producing the sequence of estimates $\{\xi(t)\}_t$ such that $\xi(t) \in \Xi$ for all $t \in \mathbb{Z}_{0+}$ according to Assumption 1, design the sequence of predicted convex combination vectors ξ_t^N, the terminal cost $\mathcal{P}(\xi)$, the robust terminal set $\mathcal{X}_N$, and the robust constraint set $\mathcal{C}_{xu}$ in (3) so that the IAMPC controller that at any $t \in \mathbb{Z}_{0+}$ solves (3) and applies $u(t) = u^*_{0|t}$ achieves: (i) ISS of the closed-loop with respect to $\tilde{\xi}_{0|t} = \bar{\xi} - \xi_{0|t}$, (ii) robust constraint satisfaction even when $\tilde{\xi}_{0|t} \neq 0$, (iii) guaranteed convergence of the runtime numerical algorithms and computational load comparable to a (non-adaptive) MPC. □

The rationale for seeking ISS in Problem 1 is that, when the unknown parameters do not change or change slowly, a "well designed" estimator will eventually converge, and hence, the closed-loop becomes AS. However, in Problem 1 ISS and constraint satisfaction hold regardless of the estimator convergence. The expansion term in the ISS Lyapunov function captures the dependency of the closed-loop performance on estimation error while allowing for an independent estimator design.

Consider the linear parameter-varying (LPV) system

$$x(t+1) = \sum_{i=1}^{\ell} [\xi(t)]_i A_i x(t) + Bu(t), \tag{4}$$

where for all $t \in \mathbb{Z}_+$, $\xi(t) \in \Xi$, the parameter-dependent (linear) control law

$$u = \kappa(\xi) x = \left(\sum_{i=1}^{\ell} [\xi]_i K_i \right) x, \tag{5}$$

and the parameter-dependent (quadratic) function

$$\mathcal{V}_\xi(x) = x' \mathcal{P}(\xi) x = x' \left(\sum_{i=1}^{\ell} [\xi]_i P_i \right) x, \tag{6}$$

where $P_i > 0$, $i \in \mathbb{Z}_{[1,\ell]}$.

Definition 5 ([14]): function (6) such that $\mathcal{V}_{\xi(t+1)}(x(t+1)) - \mathcal{V}_{\xi(t)}(x(t)) \leq 0$, for all $\xi(t), \xi(t+1) \in \Xi$, where equality holds only if $x = 0$, is a *parameter-dependent Lyapunov function* (pLF) for (4) in closed-loop with (5). $\square$

By [5], [13], [14], given $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ $Q, R > 0$, any solution $G_i, S_i \in \mathbb{R}^{n\times n}$, $S_i > 0$, $E_i \in \mathbb{R}^{m\times n}$, $i \in \mathbb{Z}_{[1,\ell]}$, of

$$\begin{bmatrix} G_i+G_i'-S_i & (A_iG_i+BE_i)' & E_i' & G_i' \\ (A_iG_i+BE_i) & S_j & 0 & 0 \\ E_i & 0 & R^{-1} & 0 \\ G_i & 0 & 0 & Q^{-1} \end{bmatrix} > 0, \ \forall i, j \in \mathbb{Z}_{[1,\ell]}. \tag{7}$$

is such that (5), (6) where $P_i = S_i^{-1}$, $K_i = E_i G_i^{-1}$, $i \in \mathbb{Z}_{[1,\ell]}$, satisfy

$$\begin{aligned} &\mathcal{V}(x(t+1), \xi(t+1)) - \mathcal{V}(x(t), \xi(t)) \leq \\ &-x(t)'(Q + \kappa(\xi(t))'R\kappa(\xi(t)))x(t), \ \forall \xi(t), \xi(t+1) \in \Xi \end{aligned} \tag{8}$$

for the closed-loop (4), (5).

Assumption 2: For the given A_i, $i \in \mathbb{Z}_{[1,\ell]}$, B, Q, R, (7) admits a feasible solution

The LMI (7) is a relaxation of those in [4]–[6]. Thus, Assumption 2 is implied by the existence of an (unconstrained) stabilizing linear control law for (1a). Indeed, if the uncertainty is too large, (7) may be infeasible. However, (7) is used here for design, and hence such situation will be recognized and can be corrected before controller execution. By using (7) only for design, the proposed method solves online only QPs, which makes it feasible also for applications with fast dynamics and low-cost microcontrollers [2], [3].

Due to limited space, in what follows the full proofs are omitted, and their key steps are briefly discussed.

III. Unconstrained IAMPC: ISS Property

We start with the unconstrained case, $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^m$.

A. Stability with parameter prediction along the horizon

Consider first the case $\xi_{k|t} = \bar{\xi}(t+k)$, $k \in \mathbb{Z}_{[0,N]}$, where it is possible that $\xi_{k_1|t} \neq \xi_{k_2|t}$, for $k_1, k_2 \in \mathbb{Z}_{[0,N]}$. This amounts to controlling an LPV system with N steps of parameter preview, but no information afterwards.

Lemma 1: Let Assumption 2 hold and consider (4) and the MPC that at $t \in \mathbb{Z}_{0+}$ solves (3) where $\mathcal{X}_N = \mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^m$, $\mathcal{C}_{xu} = \mathbb{R}^{n+m}$, $\mathcal{U} = \mathbb{R}^m$, $\xi_{k|t} = \bar{\xi}(t+k)$, and $\mathcal{P}(\xi)$, $\kappa(\xi)$ are from (7). Then, the origin is AS for the closed loop with domain of attraction $\mathbb{R}^n$ for every sequence $\{\bar{\xi}(t)\}_t$, such that $\bar{\xi}(t) \in \Xi$, for all $t \in \mathbb{R}_{0+}$. $\square$

The proof of Lemma 1 is obtained by adapting the proofs for unconstrained MPC extended to time-varying systems, see, [1, Sec.2.4], for the terminal cost designed as in (7).

By Lemma 1, the MPC based on (3) with perfect preview along the horizon is stabilizing. Next, we account for the effect of the parameter estimation error.

B. ISS with respect to parameter estimation error

Consider now the case relevant to Problem 1 where $\bar{\xi}(t)$ is constant, i.e., $\bar{\xi}(t) = \bar{\xi}$, for all $t \in \mathbb{Z}_{0+}$, unknown, and being estimated. Thus, $\tilde{\xi}_{0|t} = \bar{\xi} - \xi_{0|t}$ is the error in the parameter estimate, which may be time-varying, and $\tilde{\xi}_{0|t} \in \Xi(\xi_{0|t})$. The parameter estimation error induces a state prediction error

$$\varepsilon_x = \sum_{i=1}^{\ell} [\bar{\xi}]_i A_i x - \sum_{i=1}^{\ell} [\xi_{0|t}]_i A_i x = \sum_{i=1}^{\ell} [\tilde{\xi}_{0|t}]_i A_i x. \tag{9}$$

Indeed,

$$\begin{aligned} \|\varepsilon_x\| &= \left\| \sum_{i=1}^{\ell} [\tilde{\xi}_{0|t}]_i A_i x \right\| \leq \left\| \sum_{i=1}^{\ell} [\tilde{\xi}_{0|t}]_i A_i \right\| \cdot \|x\| \\ &\leq \left(\sum_{i=1}^{\ell} |[\tilde{\xi}_{0|t}]_i| \, \|A_i\| \right) \|x\| \leq \gamma_A \|\tilde{\xi}_{0|t}\|_1 \|x\| \end{aligned} \tag{10}$$

where $\gamma_A = \max_{i=1,\ldots,\ell} \|A_i\|$.

For the value function $\mathcal{V}^{\mathrm{MPC}}_{\xi^N}$ of (3), the following result is straightforward from [1].

Result 2: For every compact $\mathcal{X}_L \subseteq \mathbb{R}^n$, the value function of (3), where $\mathcal{P}(\xi)$ is designed according to (7), is Lipschitz-continuous in $x \in \mathcal{X}_L$, that is, there exists $L \in \mathbb{R}_+$ such that for every $x_1, x_2 \in \mathcal{X}_L$, $\|\mathcal{V}^{\mathrm{MPC}}_{\xi^N}(x_1) - \mathcal{V}^{\mathrm{MPC}}_{\xi^N}(x_2)\| \leq L\|x_1 - x_2\|$, for every $\xi^N \in \Xi^{N+1}$.

Result 2 follows directly from the fact that for every $\xi^N \in \Xi^{N+1}$, $\mathcal{V}^{\mathrm{MPC}}_{\xi^N}$ is piecewise quadratic [1] and hence it is Lipschitz continuous in any compact set $\mathcal{X}_L$. Thus, for any $\mathcal{X}_L \subseteq \mathbb{R}^n$ and $\xi^N \in \Xi^{N+1}$, there exists a Lipschitz parameter $L_{\xi^N} \in \mathbb{R}_+$. Since Ξ^{N+1} is compact, i.e., closed and bounded, there exists a maximum of $L_{\xi^N} \in \mathbb{R}_+$ for $\xi^N \in \Xi^{N+1}$, which gives a Lipschitz constant L for $\mathcal{V}^{\mathrm{MPC}}_{\xi^N}$.

Lemma 2: Let $\xi_{k-1|t+1} = \xi_{k|t}$, for all $k \in \mathbb{Z}_{[1,N]}$, $t \in \mathbb{Z}_{0+}$. Then, there exists $\gamma_L > 0$, such that for every $x \in \mathcal{X}_L$,

$$\begin{aligned} \mathcal{V}^{\mathrm{MPC}}_{\xi^N_{t+1}}(x(t+1)) \leq & \mathcal{V}^{\mathrm{MPC}}_{\xi^N_t}(x(t)) - \lambda_{\min}(Q)\|x(t)\|^2 \\ & + \gamma_L \|\tilde{\xi}_{0|t}\|_1 \|x(t)\|. \end{aligned} \tag{11}$$

Theorem 1: Let Assumptions 1, 2 hold, and $\xi_{k|t+1} = \xi_{k+1|t}$, for all $k \in \mathbb{Z}_{[0,N-1]}$, and all $t \in \mathbb{Z}_{0+}$. For the MPC that at any step solves (3) where $\mathcal{P}(\xi)$ is designed according to (7), $\mathcal{X}_N = \mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^m$, $\mathcal{C}_{xu} = \mathbb{R}^{n+m}$, $\mathcal{U} = \mathbb{R}^m$, $\mathcal{V}^{\mathrm{MPC}}_{\xi^N}(x)$ is an ISS-Lyapunov function with respect to the estimation error $\tilde{\xi}_{0|t} = \bar{\xi} - \xi_{0|t} \in \tilde{\Xi}(\xi_{0|t})$ for (1) in closed loop with the MPC based on (3) in any $\mathcal{X}_\eta \subseteq \mathcal{X}_L$, where $\mathcal{X}_\eta$ is RPI with respect to $\tilde{\xi}_{0|t}$ for the closed loop. $\square$

The proof of Lemma 2 follows from Lemma 1 and Lipschitz continuity. That of Theorem 1 follows from Lemma 2, compactness of $\mathcal{X}_L$ and the properties of the norms.

IV. Constrained IAMPC: Robust Constraints

In Section III we obtained an unconstrained IAMPC that guarantees ISS with respect to $\tilde{\xi}_{0|t}$. Next, we consider constrained IAMPC, i.e., $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^m$.

Assumption 3: $\mathcal{X}, \mathcal{U}$ are compact polyhedra with $0 \in \mathrm{int}(\mathcal{X}), 0 \in \mathrm{int}(\mathcal{U})$.

Under Assumption 3, we design $\mathcal{X}_N$ such that the LPV system (4) with perfect preview along the prediction horizon, i.e., $\xi_{k|t} = \bar{\xi}(t+k)$ for all $k \in \mathbb{Z}_{[0,N]}$, recursively satisfies the constraints. Then, we design $\mathcal{C}_{xu}$ to enforce constraint satisfaction when $\tilde{\xi}_{0|t} \neq 0$.

A. Terminal set design for nominal terminal constraint

Consider (4) where $\xi(t)$ is known at $t \in \mathbb{Z}_{0+}$ and the control law (5) resulting in the closed-loop LPV system

$$x(t+1) = \sum_{i=1}^{\ell} [\xi(t)]_i (A_i + BK_i) x(t). \tag{12}$$

The trajectories of (12) are contained in those of the pLDI

$$x(t+1) \in \text{co}\{(A_i + BK_i)x(t)\}_{i=1}^{\ell}. \tag{13}$$

For (13) in closed loop with (5) designed by (7) subject to (1b), in [13] it was shown that the maximum constraint admissible set $\mathcal{X}^{\infty} \subseteq \bar{\mathcal{X}}$, where $\bar{\mathcal{X}} = \{x \in \mathcal{X} : \kappa(\xi)x \in \mathcal{U}, \forall \xi \in \Xi\}$ is polyhedral, finitely determined and has non-empty interior with $0 \in \text{int}(\mathcal{X}^{\infty})$. $\mathcal{X}^{\infty}$ is RPI for (12) for all $\xi \in \Xi$, and is the limit of a sequence of backward reachable sets. Let $\mathcal{X}_{xu}$ be a given set of feasible states and inputs $\mathcal{X}_{xu} \subseteq \mathcal{X} \times \mathcal{U}$, $0 \in \text{int}(\mathcal{X}_{xu})$, and let

$$\begin{aligned} \mathcal{X}^{(0)} &= \{x : \ (x, K_i x) \in \mathcal{X}_{xu}, \ \forall i \in \mathbb{Z}_{[1,\ell]}\} \\ \mathcal{X}^{(h+1)} &= \{x : \ (A_i + BK_i)x \in \mathcal{X}^{(h)}, \ \forall i \in \mathbb{Z}_{[1,\ell]}\} \cap \mathcal{X}^{(h)} \\ \mathcal{X}^{\infty} &= \lim_{h \to \infty} \mathcal{X}^{(h)}. \end{aligned} \tag{14}$$

Due to the finite determination of $\mathcal{X}^{\infty}$ there exists a finite $\bar{h} \in \mathbb{Z}_{0+}$ such that $\mathcal{X}^{(\bar{h}+1)} = \mathcal{X}^{(\bar{h})} = \mathcal{X}^{\infty}$.

Lemma 3: Consider (4) and the MPC that at $t \in \mathbb{Z}_{0+}$ solves (3) where $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{U} \subset \mathbb{R}^m$, $\mathcal{C}_{xu} = \mathbb{R}^{n+m}$, $\xi_{k|t} = \bar{\xi}(t+k)$, $\mathcal{P}(\xi)$, $\kappa(\xi)$ are designed according to (7) and $\mathcal{X}_N = \mathcal{X}^{\infty}$, where $\mathcal{X}^{\infty}$ is from (14). At a given $t \in \mathbb{Z}_{0+}$, let $x(t) \in \mathcal{X}$, $\xi_t^N \in \Xi^{N+1}$ be such that (3) is feasible. Then, (3) is feasible for any $\tau \geq t$, i.e., $\mathcal{X}_f(\xi^N) = \{x \in \mathcal{X} :$ (3) feasible for $x_{0|t} = x, \ \xi_{k|t} = \xi_k \in \Xi, \ k \in \mathbb{Z}_{[0,N]}\}$ is a PI set, and the origin is AS in $\mathcal{X}_f(\xi^N)$. □

The proof of Lemma 3 is based on proving that $\mathcal{X}_f(\xi^N)$ is PI, due to the terminal set $\mathcal{X}_N$ from (14), thus ensuring recursive feasibility. Combined with Theorem 1, this proves AS in $\mathcal{X}_f(\xi^N)$.

B. Robust constraints design

In order to ensure robust constraint satisfaction in the presence of parameter estimation error we design the constraint (3e) from a RCI set for the pLDI (2). Based on Definition 2, let $\mathcal{C} \subseteq \mathcal{X}$ be a convex set such that for any $x \in \mathcal{C}$ there exists $u \in \mathcal{U}$ such that $A_i x + Bu \in \mathcal{C}$ for all $i \in \mathbb{Z}_{[1,\ell]}$. Given $\mathcal{C}$, we design $\mathcal{C}_{xu}$ in (3e) as

$$\mathcal{C}_{xu} = \{(x, u) \in \mathcal{C} \times \mathcal{U}, \ A_i x + Bu \in \mathcal{C}, \ \forall i \in \mathbb{Z}_{[1,\ell]}\}, \tag{15}$$

that is, the state-input pairs that result in states within the RCI set for any vertex system of the pLDI (2).

Lemma 4: Consider (3) where $\mathcal{X}_N = \mathbb{R}^n$, and $\mathcal{C}_{xu}$ in (3e) is defined by (15). If $x(t) \in \mathcal{C}$, (3) is feasible for all $\tau \geq t$, for any $\xi_\tau^N \in \Xi^{N+1}$ and any $\tilde{\xi}_{0|\tau} \in \tilde{\Xi}(\xi_{0|\tau})$. □

The proof of Lemma 4 follows from the convexity of $\mathcal{C}_{xu}$, and the pLDI update equation (2).

$\mathcal{C}$ can be computed as the maximal RCI set for (2) from the sequence [15],

$$\mathcal{C}^{(0)} = \mathcal{X}, \tag{16a}$$

$$\begin{aligned} \mathcal{C}^{(h+1)} = \{x : \ &\exists u \in \mathcal{U}, \\ &A_i x + Bu \in \mathcal{C}^{(h)}, \ \forall i \in \mathbb{Z}_{[1,\ell]}\} \cap \mathcal{C}^{(h)}. \end{aligned} \tag{16b}$$

The maximal RCI set in $\mathcal{X}$ is the fixpoint of (16), i.e., $\mathcal{C}^{\infty} = \mathcal{C}^{(\bar{h})}$ such that $\mathcal{C}^{(\bar{h}+1)} = \mathcal{C}^{(\bar{h})}$, and is the largest set within $\mathcal{X}$ that can be made invariant for (2) with inputs in $\mathcal{U}$.

To guarantee satisfaction of the terminal constraint when $(x_{k|t}, u_{k|t}) \in \mathcal{C}_{xu}$ is imposed in (3), $(x, \kappa(\xi)x) \in \mathcal{C}_{xu}$ for every $x \in \mathcal{X}_N$, $\xi \in \Xi$ must hold, and the horizon N must be selected such that for every $x \in \mathcal{C}$ and $\xi^N \in \Xi^{N+1}$, there exists $[u(0) \dots u(N-1)]$ such that for (4) with $x(0) = x$, $\xi(k) = \xi_k$ for all $k \in \mathbb{Z}_{[0,N]}$, $(x(k), u(k)) \in \mathcal{C}_{xu}$ for all $k \in \mathbb{Z}_{[0,N-1]}$, and $x(N) \in \mathcal{X}_N$. Let

$$\begin{aligned} \mathcal{S}^{(0)} &= \mathcal{X}_N, \\ \mathcal{S}_i^{(h+1)} &= \{x \in \mathcal{X} : \ \exists u \in \mathcal{U}, \ A_i x + Bu \in \mathcal{S}^{(h)}\}, \\ \mathcal{S}^{(h+1)} &= \bigcap_{i=1}^{\ell} \mathcal{S}_i^{(h+1)}. \end{aligned} \tag{17}$$

The set $\mathcal{S}^{(h)}$ is such that for any $x(0) \in \mathcal{S}^{(h)}$, given any $\xi^{h-1} \in \Xi^h$, there exists a sequence $[u(0) \dots u(h-1)]$ such that for (4) with $x(0) = x$ and $\xi(k) = \xi_k$ for all $k \in \mathbb{Z}_{[0,N]}$, $(x(k), u(k)) \in \mathcal{C}_{xu}$ and $x(h) \in \mathcal{X}_N$.

Theorem 2: Consider (3), let $\bar{h} \in \mathbb{Z}_{0+}$ be such that $\mathcal{C}^{(\bar{h}+1)} = \mathcal{C}^{(\bar{h})} = \mathcal{C}$ in (16), and let $\mathcal{C}_{xu}$ be defined by (15). Let $\mathcal{X}_N = \mathcal{X}^{\infty}$ from (14), where $\mathcal{X}_{xu} = \mathcal{C}_{xu}$, and $N \in \mathbb{Z}_{0+}$ be such that $\mathcal{S}^{(N)} \supseteq \mathcal{C}$. If $x(t) \in \mathcal{C}$ at $t \in \mathbb{Z}_{0+}$, and $\xi_\tau^N \in \Xi^{N+1}$, $\tilde{\xi}_{0|\tau} \in \tilde{\Xi}(\xi_{0|\tau})$ for all $\tau \geq t$, (3) is feasible for all $\tau \geq t$. If there exists $t \in \mathbb{Z}_{0+}$ such that $\xi_{k|\tau} = \bar{\xi}(\tau + k)$ for all $\tau \geq t$, $k \in \mathbb{Z}_{[0,N]}$, (1) in closed-loop with the MPC that solves (3) is also AS in $\mathcal{C}$. □

The proof of Theorem 2 follows from combining the results of Lemma 3 with the robust invariance of $\mathcal{C}$ and the fact that N is such that $\mathcal{C} \subseteq \mathcal{S}^{(N)}$. In the construction of $\mathcal{S}^{(h)}$, i.e., (17), the parameter sequence ξ^h is known since the terminal set is enforced with respect to the nominal dynamics. The robust invariance of $\mathcal{C}$ and the choice of N such that $\mathcal{S}^{(N)} \supseteq \mathcal{C}$ guarantee that, even in presence of a parameter estimation error, $\mathcal{X}_N$ can be reached in N steps.

Conditions for existence of $\mathcal{C}^{\infty}$ are related to the existence of a nonlinear stabilizing law for (1a), and are discussed in details in [15]. Theorem 2 ensures robust feasibility of (3), robust satisfaction of (1b), and nominal asymptotic stability, i.e., if there exists $t \in \mathbb{Z}_{0+}$ such that $\tilde{\xi}_{k|\tau} = 0$, for all $\tau \geq t$, $k \in \mathbb{Z}_{[0,N]}$, the closed loop is AS.

V. Indirect Adaptive MPC: Complete Algorithm

The last design element in (3) is the construction of the parameter prediction vector ξ_t^N.

Since $\bar{\xi}$ in (1) is assumed to be constant or slowly varying, an obvious choice would be $\xi_{k|t} = \xi(t)$, for all $k \in \mathbb{Z}_{[0,N]}$, for all $t \in \mathbb{Z}_{0+}$. However, this choice violates the assumption of Theorem 1 (and implicitly those of Lemmas 1 and 3) that requires $\xi_{k|t+1} = \xi_{k+1|t}$, for all $k \in \mathbb{Z}_{[0,N-1]}$, $t \in \mathbb{Z}_{0+}$. Such an assumption is required because if the entire parameter prediction vector ξ_t^N suddenly changes, the value function $\mathcal{V}_N^{\mathrm{MPC}}$ may not be decreasing.

Thus, we introduce a N-step delay in the parameter prediction,

$$\xi_{k|t} = \xi(t - N + k), \ \forall k \in \mathbb{Z}_{[0,N]}. \tag{18}$$

Due to (18), at each time t, the new estimate is added as last element of ξ_t^N, i.e., $\xi_{N|t} = \xi(t)$ and $\xi_{k|t} = \xi_{k+1|t-1}$, for all $k \in \mathbb{Z}_{[0,N-1]}$, $t \in \mathbb{Z}_{0+}$. We can now state the complete properties of IAMPC.

Theorem 3: Let Assumptions 1–3 hold. Consider (1), where $\bar{\xi} \in \Xi$, in closed loop with the IAMPC controller that at every $t \in \mathbb{Z}_{0+}$ solves (3), where $\mathcal{P}(\xi)$ defined by (6) and $\kappa(\xi)$ defined by (5) are from (7), $\mathcal{C}$, $\mathcal{X}_N$, and N are designed according to Theorem 2 and $\xi_t^N \in \Xi^{N+1}$ is obtained from (18). If for some $t \in \mathbb{Z}_{0+}$, $x(t) \in \mathcal{C}$, the closed-loop satisfies (1b), and (3) is recursively feasible for any $\tau \geq t$. Furthermore, the closed loop is ISS in the RPI set $\mathcal{C}$ with respect to $\tilde{\xi}_{0|t} = \bar{\xi} - \xi_{0|t}$, i.e., the N-steps delayed estimation error $\tilde{\xi}_{0|t} = \bar{\xi} - \xi(t - N)$.

Proof: The proof follows by combining Theorem 1 with Theorem 2. By Theorem 2, $\mathcal{C}$ is RCI, and if $x(t) \in \mathcal{C}$, (3) is feasible for all $\tau \geq t$, for any $\xi_\tau^N \in \Xi^{N+1}$ that satisfies (18), since (18) implies that $\xi_{k|\tau} = \xi_{k+1|\tau-1}$, for all $k \in \mathbb{Z}_{[0,N-1]}$. Thus, by (15) enforced in (3), $\mathcal{C} \subseteq \mathcal{X}$ is a compact RPI for the closed-loop system, and hence (1b) is satisfied for all $\tau \geq t$. Since $\mathcal{V}_{\xi^N}^{\mathrm{MPC}}$ is piecewise quadratic for every $\xi^N \in \Xi^{N+1}$, by taking $\mathcal{X}_\eta = \mathcal{X}_L = \mathcal{C}$, which is RPI for the closed-loop system and compact since $\mathcal{C} \subseteq \mathcal{X}$, the existence of a Lipschitz constant L is guaranteed, according to Result 2. Hence, Theorem 1 holds within $\mathcal{C}$, proving ISS with respect to $\tilde{\xi}_{0|t} = \bar{\xi} - \xi_{0|t} = \bar{\xi} - \xi(t - N)$, i.e., the delayed estimation error. ∎

Based on Theorem 3, from any initial state $x(t) \in \mathcal{C}$, the closed-loop system robustly satisfies the constraints for any admissible estimation error, and the expansion term in the ISS Lyapunov function is proportional to the norm of the delayed parameter estimation error. Thus, if the parameter estimate converges at time t^* and such value is maintained for all $t \geq t^*$, for all $t \geq t^* + N$, $\tilde{\xi}_t^N = 0$ and hence the closed-loop is AS. Finally, note that at runtime, the IAMPC only solves a QP as a standard (non-adaptive) linear MPC. Thus, based on Theorem 3 we can state the following.

Result 3: The IAMPC designed according to Theorem 3 solves Problem 1.

The ISS property established in Theorem 3 implies that when the estimator converges to the true parameter value the closed-loop becomes AS. But ISS also ensures that, even if the estimate never converges, the ultimate bound on the state is proportional to the estimation error. Thus, ISS allows to state properties that hold regardless of the convergence of the estimator and do not require a specific choice for the estimator design. On the other hand, it is required for the estimator to provide $\xi(t) \in \Xi$, for all $t \in \mathbb{Z}_{0+}$, as per

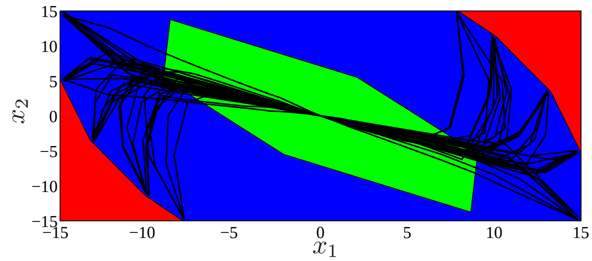

Fig. 1. IAMPC simulations on the numerical example. Trajectories (black), $\mathcal{X}^\infty$ (green), $\mathcal{C}$ (blue), $\mathcal{X}$ (red).

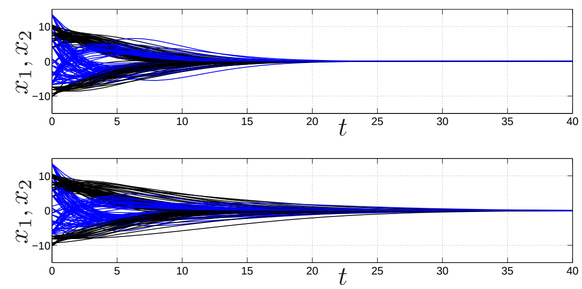

Fig. 2. Simulations with fast ($\varsigma = 1/2$, upper plot) and slow ($\varsigma = 1/20$, lower plot) parameter estimator, $[x]_1$ blue, $[x]_2$ black.

Assumption 1. To enforce Assumption 1 one can always design an estimator that produces the unconstrained estimate $\varrho \in \mathbb{R}^\ell$, and provides to the IAMPC the projection onto Ξ, i.e., $\xi = \mathrm{proj}_\Xi(\varrho)$. By using $\varrho \in \mathbb{R}^\ell$ in the estimator update and providing $\xi = \mathrm{proj}_\Xi(\eta)$ to the controller, this amounts to a standard (unconstrained) estimator with an output nonlinearity. Thus, the convergence conditions are similar to those for standard estimators, in particular, identifiability and persistent excitation [11], [12]. As for identifiability, it is worth noticing that the true value of the parameter $\bar{\xi}$ may not be uniquely defined due to the polytopic representation (1). While not proved here due to limited space, in this case a slightly modified ISS Lyapunov function can be provided, related to the smallest error between the current estimate and all the convex parameter vectors corresponding to the actual system matrices. Thus, it is not necessary to reconstruct a specific value $\bar{\xi}$, but rather any value that is associated to the actual system matrices.

VI. Numerical Simulations

We consider (1), where $\ell = 5$, and $A_1 = \left[\begin{smallmatrix} 1 & 0.2 \\ 0 & 1 \end{smallmatrix}\right]$, $A_2 = 1.1 \cdot A_1$, $A_3 = 0.6 \cdot A_1$, $A_4 = \left[\begin{smallmatrix} 0.9 & 0.3 \\ 0.4 & 0.6 \end{smallmatrix}\right]$ $A_5 = \left[\begin{smallmatrix} 0.95 & 0 \\ 0.8 & 1.02 \end{smallmatrix}\right]$, and $B = \left[\begin{smallmatrix} -0.035 & -0.905 \end{smallmatrix}\right]'$. While being only constructed from 2^{nd} order systems, this example is challenging because some of the dynamics are stable and some unstable, and the system matrices are in some cases significantly different. The constraints are defined by (1b), where $\mathcal{X} = \{x \in \mathbb{R}^2 : |[x]_i| \leq 15,\ i = 1, 2\}$, $\mathcal{U} = \{u \in \mathbb{R} : \ |[u]| \leq 10\}$. We have implemented a simple estimator that computes the least squares solution $\varrho(t)$ based on past data window of

N_m steps and applies a first order filter on the projection of $\varrho(t)$ onto Ξ, i.e., $\xi(t+1) = (1-\varsigma)\xi(t) + \varsigma \cdot \mathrm{proj}_{\Xi}(\varrho(t))$, where $\varsigma \in \mathbb{R}_{(0,1)}$, and $[\xi(0)]_i = 1/\ell$, $i \in \mathbb{Z}_{[1,\ell]}$. Such simple estimator satisfies Assumption 1 and requires only solving a (small) QP. In the simulations we use the QP solver in [16] for both estimation and control computation. We design the controller according to Theorem 2, where $\mathcal{C} = \mathcal{C}^\infty$, and we select $N_m = 3$ and $N = 8$, which is the smallest value such that $\mathcal{S}^{(N)} \supseteq \mathcal{C}^\infty$ by (17). Figure 1 shows the simulations where the initial conditions are the vertices of $\mathcal{C}$ and for each initial condition, 4 different simulations with different (random) values of $\bar{\xi} \in \Xi$ are executed. Figure 2 compares the cases where $\varsigma = 1/2$ and $\varsigma = 1/16$, i.e., fast versus slow estimation, thus showing the impact of the estimation error on the closed-loop behavior.

As an additional case study, we consider the compressor control of a variable refrigerant flow air conditioner (VRF-AC). The model is a simplification of that in [17], obtained by first principles and data, where the valve and the fan speeds are kept constant. The resulting 4^{th} order model is linearized around the setpoint (x_{ss}, u_{ss}) where the state coordinates are chosen as $x = [T_r \ T_d \ T_e \ \zeta]$, T_r[deg] is the room temperature, T_e[deg] is the evaporating temperature, T_d[deg] is the compressor discharge temperature, ζ is a nonphysical state related to internal conditions of the air conditioner, and the control input is the compressor frequency $u = C_F$[Hz]. The setpoint is $x_{\mathrm{ss}} = [22 \ 9 \ 72 \ 62]$, $u_{\mathrm{ss}} = 45$. The controller must enforce upper and lower bounds on state, $\overline{x} = [3 \ 5 \ 5 \ 20]$, $\underline{x} = -[0.5 \ 3 \ 10 \ 20]$, and input $\overline{u} = -\underline{u} = 20$. We consider uncertainty in the thermal mass of the room by $\pm 50\%$ and in the efficiency of the energy transfer from the evaporator to the room by $\pm 20\%$, obtaining (1) with $\ell = 4$. We design the IAMPC and the parameter estimator with $N = 8$, $T_s = 1$min, $N_m = 2$. The simulation results from multiple initial conditions for different realizations of the uncertainty are shown in Figure (3). As regards implementability of IAMPC, it is worth remarking that the QP solver [16], as opposed to LMI solvers, is feasible for implementation on the air conditioner microcontroller [17].

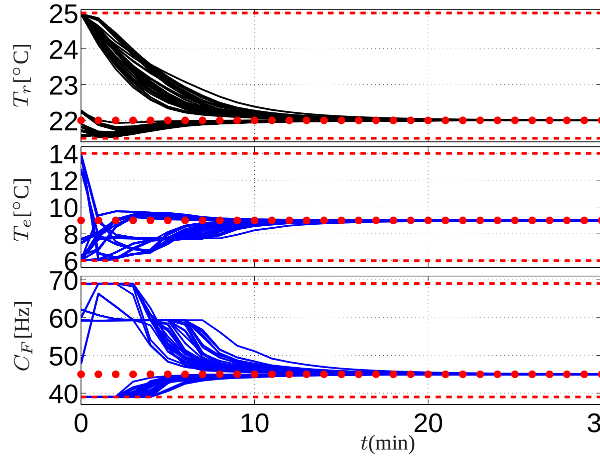

Fig. 3. IAMPC control of air conditioning system. Time trajectories (solid) constraints (dash), setpoint (dot).

VII. Conclusions and Future Work

We have proposed an indirect adaptive MPC that guarantees robust constraint satisfaction, recursive feasibility, and ISS with respect to the parameter estimation error, and has computational requirements similar to standard MPC. The IAMPC designs exploits a terminal cost designed as a pLF, a robust PI terminal set, and a RCI set for ensuring robust constraint satisfaction. The control design allows to choose any parameter estimation algorithm as long as the estimate of the parameter used for prediction lies in a specified set, which can be obtained by projecting the unconstrained estimate.

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