

AutoBayes: Automated Machine Learning with Bayesian Graph Exploration for Nuisance-Robust Inference



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- Part I: Trends of machine learning
- Part II: Adversarial learning for nuisance-robust data analysis
- Part III: Meta learning: Automated machine learning (AutoML)
 - Automated architecture and hyperparameter tuning
- Part IV: AutoBayes
 - Bayesian inference graph modeling
 - Bayes Ball algorithm
- Part V: Ensemble learning
- Summary



Answering... What is best DNN architecture?







• Gartnar's Hype Cycle for Emerging Technologies, 2018 July



MITSUBISHI Changes for the Better Changes for the Better

• Gartnar's Hype Cycle for Emerging Technologies, 2019 August



MITSUBISHI Changes for the Better Changes for the Better

- Deep learning = fancy name of multi-layer perceptron neural networks.
 - 2006 Hinton: Many layers, layer-wise pre-training, massive data sets
- Massively parallel computation
 - Driver: graphic processor units, tensor processor units ...
- Variants:
 - Deep belief networks
 - Deep convolutional networks
 - Deep recurrent networks
 - Deep Boltzmann machines
 - Deep autoencoder





Deep Boltzmann Machine

Deep Belief Network





Convolutional Networks



Recurrent Networks



• Audio & Visual Applications







motor scooter	leopard	
motor scooter	leopard	
go-kart	jaguar	
moped	cheetah	
bumper car	snow leopard	
golfcart	Egyptian cat	









"man in black shirt is playing guitar."



AI Surpassing Human-Level Performance





Moore's Law: Exponential Growth in Applications

• Hit count of articles per year in GoogleScholar; Wireless Communication applications



Moore's Law: Exponential Growth in Applications II

• Hit count of articles per year in GoogleScholar; **Optical Communication** applications













- Joint work with Prof. Deniz Erdogmus (Northeastern Univ.)
- 2015 Ruhi Mahajan
 - Authentication (EMBC)
- 2016 Fernando Quivira
 - Probabilistic GMM+LSTM (BHI)
- 2017 Chun-Shu Wei
 - Few-shot learning (NER)
- 2018 Ozan Ozdenizci
 - Adversarial VAE (NER, SPL, Access)
- 2019 Mo Han
 - Complementary adversarial (EMBC, SPL)
 - Rateless soft disentangling (JBHI)
- 2020 Andac Demir
 - AutoBayes (Access)
 - Graph EEG net (EMBC)











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Adversarial Learning





- Generative Adversarial Networks (GAN) [Goodfellow et al, 2014]
 - Train two competing neural networks
 - Generator learns to fake images by trying to fool Discriminator



• Competition between counterfeiters and police \Rightarrow better fake money



• Nvidia GAN Results [Karras et al, 2018]



Realistic Fake Faces youtube:XOxxPcy5Gr4 youtube:kSLJriaOumA





• CycleGAN [Zhu et al, 2017]





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MITSUBISHI **GAN Zoo – Lots of Adversarial Networks** Changes for the Better

Many different ways to adversarially combine networks





Learning Nuisance-Invariant Data Representations



- Objective: extract invariant representations (features)
 - Remove nuisance variations, sensitive attributes
 - Motivation: transferability, generalizability, robustness, privacy, fairness
- Autoencoder model: data $\mathbf{x} \to \operatorname{Enc} \to \operatorname{latent} \mathbf{z} \to \operatorname{Dec} \to \widehat{\mathbf{x}}$
 - General purpose data representations z
 - Can also support translation, feature/style transfer

ating	Price
5 ★★★★★ (225)	\$11-30



- VAE introduced by [Kingma & Welling, 2014] with conditional extension [Sohn et al, 2015]
- Learn CVAE model: $(\mathbf{x}, \mathbf{s}, \mathbf{z}) \sim p_{\theta}(\mathbf{x} | \mathbf{z}, \mathbf{s}) p(\mathbf{s}) p(\mathbf{z})$
 - x raw data features

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- s nuisance variations (conditioning variable)
- z latent (unobserved) representation
- Invariance: model explicitly specifies independence between \boldsymbol{s} and \boldsymbol{z}
- Generative model $p(\mathbf{x}|\mathbf{z}, \mathbf{s})$ from appropriate parametric family
- Convenient latent model $p(\mathbf{z}) = N(\mathbf{0}; \mathbf{I})$
- Nuisance model $p(\mathbf{s})$ arbitrary (not used for training)







- Decoder: generative model $p_{\theta}(\mathbf{x}|\mathbf{z}, \mathbf{s})$
- Encoder: variational posterior $q_{\phi}(\mathbf{z}|\mathbf{x}, \mathbf{s})$

• In principle, $q_{\phi}(\mathbf{z}|\mathbf{x}, \mathbf{s}) \to p_{\theta}(\mathbf{z}|\mathbf{x}, \mathbf{s})$ and hence $\mathbf{z} \perp \mathbf{s}$

• However, in practice, invariance $(I(\mathbf{s}; \mathbf{z}) = 0)$ needs to be enforced

 $\max_{\theta,\phi} \mathcal{L}(\theta,\phi) - \lambda I(\mathbf{s};\mathbf{z})$

VAE Training with Adversarial Censoring



- Decoder: generative model $p_{\theta}(\mathbf{x}|\mathbf{z},\mathbf{s})$
- Encoder: variational posterior $q_{\phi}(\mathbf{z}|\mathbf{x}, \mathbf{s})$

In principle, $q_{\phi}(\mathbf{z}|\mathbf{x}, \mathbf{s}) \to p_{\theta}(\mathbf{z}|\mathbf{x}, \mathbf{s})$ and hence $\mathbf{z} \perp \mathbf{s}$

• However, in practice, invariance $(I(\mathbf{s}; \mathbf{z}) = 0)$ needs to be enforced

$$\max_{\theta,\phi} \mathcal{L}(\theta,\phi) - \lambda I(\mathbf{s};\mathbf{z}) \Rightarrow \max_{\theta,\phi} \min_{\psi} \mathcal{L}(\theta,\phi) \underbrace{-\lambda \mathbb{E} \left[\log q_{\psi}(\mathbf{s}|\mathbf{z}) \right]}_{\geq -\lambda(I(\mathbf{z};\mathbf{s}) - h(\mathbf{s}))}$$

• Adversary $q_{\psi}(\mathbf{s}|\mathbf{z})$ attempts to recover \mathbf{s} , approximates $I(\mathbf{s};\mathbf{z})$



• Wang, Y., Koike-Akino, T., Erdogmus, D. "Invariant Representations from Adversarially Censored Autoencoders", arxiv:1805.08097, May 2018.



- Full (•): full VAE conditioned on s, i.e. $E(\mathbf{x}, \mathbf{s})$, $D(\mathbf{z}, \mathbf{s})$
- Partial (\blacktriangle): only decoder conditioned on s, i.e. $E(\mathbf{x})$, $D(\mathbf{z}, \mathbf{s})$
- Basic (\blacksquare): no conditioning on s, i.e. $E(\mathbf{x})$, $D(\mathbf{z})$
- KL censoring alternative: use $-\gamma \text{KL}(q_{\phi}(\mathbf{z}|\mathbf{x},\mathbf{s}) \| p(\mathbf{z}))$, with $\gamma > 1$

A-CVAE for Nuisance-Robust Transfer Learning: Zero-Shot Learning MITSUBISHI Chanaes for the Better

- Cross-subject transfer learning for BCI [Ozdenizci et al, NER'19]
 - Task: motor-imagery decoding from EEG measurements
 - Subject variability is the nuisance variation suppressed
- Cross-session EEG-based biometrics [Ozdenizci et al, SPL'19]
 - Task: subject identification from EEG measurements
 - Session variability is the nuisance variation suppressed







Evolution Map: Nuisance-Invariant Feature Extraction MITSUBISHI ELECTRIC Chanaes for the Better



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- Nano-photonic device design
- Privacy preserving
- Localization
- Image sensing
- Speech recognition
- Face recognition

...

















- Forward, recursive, convolutional
- LSTM, GRU, Transformer
- Bottleneck, U-net, HR-net
- ResNet, loopy, clique

Aug. 2020





Output



inactive

- Loss functions: CrossEntropy, MSE, MAE, hinge, L1, ...
- Updaters: SGD, Adam, AdaDelta, AdaGrad, AdaMax, LBFGS, RMSprop, ...
- Learning rate schedulers: CosineAnnealing, Cyclic, Exponential, MultiStep, OnPlateau, ...
- Regularizations: Dropout, Batch Norm, Spectral Norm, DropConnect, StochasticDepth, ShakeDrop, Shake-Shake, ...

backward

forward

- Activations: ReLU, sigmoid, tanh, ...
- Augmentation, depth, width
- Quantization, initialization
- Pooling, unpooling, padding, ...











- Wolpert 1996: *What Does Dinner Cost?*, NASA Ames Research Center
- There is no one model that works best for every problem
- We thus shall **try multiple models** and find one that works **best for a particular problem**









Automated Machine Learning: AutoML-Zero



model = nn. <mark>Sequential</mark> (
nn.Conv2d(1,20,5),
nn.ReLU(),
nn.Conv2d(20,64,5),
nn.ReLU()
)

Human Experts Programming



AI Experts Evolutionary Programming



AutoML, Learning to Learn (L2L), Meta Learning MITSUBISHI Chanaes for the Better

- Hyperparameter exploration: Auto-Pytorch, ...
 - Bayesian optimization
 - Evolutionary optimizatior
- Architecture exploration
 - Reinforcement learning
 - Cell-based building
- Augmentation exploration

The controller (RNN)

Select on hidden state

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controller idden laye





- Automated neural architecture search may work, but having little justification
- Why linking, pruning, inserting, stacking, branching, merging, ...







AutoBayes

* Different from NASA's AutoBayes...



• Directed graph indicates marginal dependency







- Joint probability factorization:
 - X: Measurement (Image, EEG, EMG, ...)
 - Y: Label to classify (digit, mental state, ...)
 - Z: Latent variables (reduced-dimension feature, ...)
 - S: Nuisance variations (user, session, environment, ...)
 - Nobody knows true data model...

$$p(y, s, z, x) = p(y)p(s|y)p(z|s, y)p(x|z, s, y)$$

(4! Possible factorization chains)



Data generative model

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If true data model follows Markov: X – Y

p(y)p(s|y)p(z|s,y)p(x|z,s,y)

• Likelihood is independent of **S** and **Z**

p(y, s, z | x) = p(z | x) p(s | z, x) p(y | s, z, x)

- Simplest classifier model p(y|x) is sufficient
 - A-CVAE? No. Irrational to involve more functionality
 - I came up with cool complex model. No, no, no way!











• Bayesian graph yields justified inference graphs





• Conditional independence can be justified systematically with simple 10 rules



Bayes Ball 10 Rules



• Bayes-Ball finds redundant links for inference graphs





Adversarial alternating updates

$$\begin{aligned} (\theta, \psi, \eta, \mu) &= \underset{\theta, \psi, \eta, \mu}{\operatorname{arg\,min}} \mathbb{E} \Big[\mathcal{L}(\hat{y}, y) + \lambda_s \mathcal{L}(\hat{s}, s) + \lambda_x \mathcal{L}(\hat{x}', x) + \lambda_z \mathbb{KL}(z_1, z_2) - \lambda_a \mathcal{L}(\hat{s}', s) \Big], \\ (z_1, z_2) &= p_{\theta}(x), \quad \hat{y} = p_{\psi}(z_1, z_2), \quad \hat{s} = p_{\phi}(z_1), \quad \hat{x}' = p_{\mu}(z_1), \quad \hat{s}' = p_{\eta}(z_1, z_2), \end{aligned}$$



DA-VAE [Han et al. SPL'20]



• How to connect Encoder, Decoder, Classifier, Estimator, Adversary cells?

Algorithm 1 Pseudocode for AutoBayes Framework

Require: Nodes set $\mathcal{V} = [Y, X, S_1, S_2, \dots, S_n, Z_1, Z_2, \dots, Z_m]$, where Y denotes task labels, X is a measurement data, $\mathcal{S} = [S_1, S_2, \dots, S_n]$ are (potentially multiple) semi-supervised nuisance variations, and $\mathcal{Z} = [Z_1, Z_2, \dots, Z_m]$ are (potentially multiple) latent vectors Ensure: Semi-supervised training/validation datasets 1: for all permutations of node factorization from Y to X do 2: Let \mathcal{B}_n be the corresponding Bayesian graph for the permuted full-chain factorization	Automatic exploration of Bayesian graphs		
2. Let \mathcal{B}_0 be the corresponding Dayesian graph for the permuted fun-chain factorization $p(y) \cdots p(z_1 \cdots) \cdots p(x \cdots)$			
3: for all combinations of link pruning on the full-chain Bayesian graph \mathcal{B}_0 do			
4: Let \mathcal{B} be the corresponding pruned Bayesian graph	Bayes Ball to check independence		
5: Apply the Bayes-Ball algorithm on \mathcal{B} to build a conditional independency list \mathcal{I}	Buyes ball to eneck independence		
6: for all permutations of node factorization from X to Y do			
7: Let \mathcal{F}_0 be the corresponding factor graph, representing a full-chain conditional			
probability $p(\cdot x)\cdots p(z_1 \cdots)\cdots p(y \cdots,x)$			
8: Prune all redundant links in \mathcal{F}_0 based on conditional independency \mathcal{I}			
9: Let \mathcal{F} be the pruned factor graph	Inference model construction		
10: Merge the pruned Bayesian graph \mathcal{B} into the pruned factor graph \mathcal{F}			
11: Attach an adversary network \mathcal{A} to latent nodes \mathcal{Z} for $Z_k \perp S \in \mathcal{I}$			
12: Assign an encoder network \mathcal{E} for $p(\mathcal{Z} \cdots)$ in the merged factor graph			
13: Assign a decoder network \mathcal{D} for $p(x \cdots)$ in the merged factor graph			
14: Assign a nuisance indicator network \mathcal{N} for $p(\mathcal{S} \cdots)$ in the merged factor graph	Link Encoder, Decoder, Classifier,		
15: Assign a classifier network C for $p(y \cdots)$ in the merged factor graph	Estimator and Adversary Nets		
16: Adversary train the whole DNN structure with variational reparameterization to			
minimize a loss function in (11) 17. (12) (12) (12) (12) (12) (12) (12)			
1/: end for \triangleright At most $(\nu - 2)!$ combinations			
18: end for \triangleright At most $2^{ \nu (\nu =1)/2}$ combinations			
19: end for \triangleright At most $(\nu - 2)!$ combinations	Return best architectures		
20: return the best model having highest task accuracy in validation sets			



- 28x28 gray-scale images
- 10-class hand-written digits
- 60,000 training data
- 10,000 test data
- Who wrote?
- QMNIST

https://github.com/facebookresearch/qmnist

- Identical datasets of MNIST
- Extended labels (writer ID etc.)
- Training data were written by **539** NIST employees
- Testing data were written by 400 high-schoolers



- Writer ID is a nuisance: Hand-written digits may depend on the writer

Changes for the Better

• Up to 0.5% gain by nuisance-robust inference





- Stress: temperature, **heart rate**, electrodermal activity, arterial oxygen level, etc. for 4state stress level measurement
- RSVP: **EEG** for rapid serial visual presentation (RSVP) drowsiness test with 4 tasks
- MI: PhysioNet EEG Motor Imagery (MI) dataset with 4-class tasks
- ErrP: An error-related potential (ErrP) of EEG dataset in spelling task
- Faces: An implanted electrocorticography (ECoG) array dataset for visual stimulus.
- Ninapro: An electromyogram (EMG) dataset for fingers motion detection for prosthetic hands.





- Publicly available datasets
 - QMNIST: <u>https://github.com/facebookresearch/qmnist</u>
 - Stress: https://physionet.org/content/noneeg/1.0.0/
 - RSVP: <u>http://hdl.handle.net/2047/D20294523</u>
 - MI: <u>https://physionet.org/physiobank/database/eegmmidb/</u>
 - ErrP: <u>https://www.kaggle.com/c/inria-bci-challenge</u>
 - Faces: <u>https://exhibits.stanford.edu/data/catalog/zk881ps0522</u>
 - Ninapro: <u>https://zenodo.org/record/1000116#.XulppS2z3OR</u>

Datasets	Modality	Dimension	Nuisance (S)	Labels (Y)	Samples
QMNIST	Image	28×28	539	10	60,000
Stress	Temperature etc.	7	20	4	24,000
RSVP	EEG	16×128	10	4	41,400
MI	EEG	64×480	106	4	9,540
ErrP	EEG	56×250	27	2	9,180
Faces Basic	ECoG	31×400	14	2	$4,\!100$
Faces Noisy	ECoG	39×400	7	2	2,100
Ninapro	EMG	16	10	12	890,446

AutoBayes Benefit: Explore Different Models for Different Problems

• No Free Lunch Theorems: There is no one model that performs best for every dataset









- Every single model may be weak
- Combining multiple weak models may beat one strong model









- AutoML/AutoBayes explores many models
- Wasting by throwing away all weaker models?
- Employ ensemble method across explored models

850

S

 $(Y \longrightarrow X) \quad (Y \longrightarrow Z \longrightarrow X)$

- Logistic regression (LR)

Ensemble Class Scores

Stacking Base Learners

Graphical Models

– MLP

...

- Transformer (multi-head attention)



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MITSUBISHI Changes for the Better Changes for the Better

• Significant gain by ensemble learning; 1.3% gain, state-of-the-art accuracy





MITSUBISHI Changes for the Better Changes for the Better

• Significant gain by ensemble learning; Up-to 12% gain







AutoBayes Ensemble Gain



MITSUBISHI Changes for the Better Subject Variation Robustness (Stress Dataset)





- We introduced a new concept called **AutoBayes** for macro DNN architecture exploration
 - Different Bayesian graphs are explored systematically
 - Bayes Ball algorithm justifies pruning independent edges
 - Encoder, decoder, estimator, classifier, and adversary network blocks are rationally linked
- We also discussed transfer learning, adversarial learning, ensemble learning
 - Multiple architectures explored in AutoML are not wasted for final classification (as base learners)
 - Different meta learners (LR, MLP, Transformer) are evaluated to aggregate multiple models
- Demonstrated the benefit for various public physiological datasets
 - Various different modalities (image, heartrate, EEG, ECoG, EMG) and dimensionalities are considered
- Questions?
 - Contact us: koike@merl.com, yewang@merl.com
 - More details in arXiv: <u>https://arxiv.org/abs/2007.01255</u>





- <u>https://deepai.org/publication/autobayes-automated-inference-via-bayesian-graph-exploration-for-nuisance-robust-biosignal-analysis</u>
- When our arXiv was uploaded on July 2, it became top trending paper
 - Obtained 71 "likes" in 4 days
 - It was highlighted in "This Week in A.I." Newsletter on July 11

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This We	ek in A.I.
Publication of the week AutoBayes: Automated Inference via Bayesian Graph Exploration for Nuisance-Robust Biosignal Analysis Read more »	<page-header><page-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></page-header></page-header>
Molecular Latent Space Simulators	No forme and endow how is to a sume The second se

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- AutoBayes explores potential graphical models inherent to the data, rather than exploring hyperparameters of DNN blocks.
- AutoBayes offers a solid reason of how to connect multiple DNN blocks to impose conditioning and adversary censoring for the task classifier, feature encoder, decoder, nuisance indicator and adversary networks, based on an explored Bayesian graph.
- It provides a systematic automation framework to explore different inference models through the use of the Bayes-Ball algorithm and ordered factorization.
- The framework is also extensible to multiple latent representations and multiple nuisance factors.
- Besides fully-supervised training, AutoBayes can automatically build some relevant graphical models suited for semi-supervised learning.
- Ensemble learning is introduced to improve performance while AutoBayes model exploration

AutoBayes as AutoML: Macro to Micro Exploration

- In principle, AutoBayes can be applied to arbitrary number of nodes
- Splitting X, Y, Z, S macro-nodes into scalar-valued micro-nodes, AutoBayes operates like AutoML architecture search but with more theoretical justification





	(p(y)p(s y)p(z s,y)p(x z,s,y),	Model-A
	p(y)p(s y)p(z s, y)p(x z, s, y),	Model-B
	p(y)p(s y)p(z s,y)p(x z,s,y),	Model-C
	p(y)p(s y)p(z s,y)p(x z, s, y),	Model-D
	p(y)p(s y)p(z s, y)p(x z, s, y),	Model-E
$p(y, s, z, x) = \langle$	p(y)p(s y)p(z s,y)p(x z,s,y),	Model-F
	p(y)p(s y)p(z s,y)p(x z,s,y),	Model-G
	p(y)p(s y)p(z s,y)p(x z,s,y),	Model-H
	p(y)p(s y)p(z s,y)p(x z,s,y),	Model-I
	$p(y)p(s y)p(z_1 s,y)p(z_2 z_1,s,y)p(x z_2,z_1,s,y),$	Model-J
	$p(y)p(s y)p(z_1 s,y)p(z_2 z_1,s,y)p(x z_2,z_1,s,y),$	Model-K

 $\overline{(s)}$

 Z_1

S

(i) Model Jz

(c) Model Ez

Y

(Y)

 $Z \rightarrow (Y)$

 $Z_2 \longrightarrow (Y$

(j) Model Js

(d) Model Es

(s)

 Z_1

(s)

(Y)

(e) Model Fz

 Z_2

Y

(k) Model Kz

Z

(1) Model Ks

(f) Model Fs

- Slash-cancelled factors from the full-chain case explicitly indicate independence.
- Conditional independence enables pruning links in the inference factor graphs.

$$p(y, z_1, z_2, s|x) = \begin{cases} p(z_1, z_2|x)p(y, s|z_1, z_2, \not{x}), & z/ys \\ p(z_1, z_2|x)p(s|z_1, \not{x}, \not{x})p(y|\not{s}, \not{x}, z_2, \not{x}), & Z-\text{Inference} \\ p(z_1|x)p(z_2|z_1, x)p(s|z_1, \not{x}, \not{x})p(y|\not{s}, \not{x}, z_2, \not{x}), & z2/z1/s/y \\ p(z_2|x)p(z_1|z_2, x)p(s|z_1, \not{x}, \not{x})p(y|\not{s}, \not{x}, z_2, \not{x}), & z1/z2/s/y \\ p(z_1|x)p(s|z_1, x)p(z_2|s, z_1, x)p(y|\not{s}, \not{x}, z_2, \not{x}), & z2/s/z1/y \\ p(s|x)p(z_1|s, x)p(z_2|s, z_1, x)p(y|\not{s}, \not{x}, z_2, \not{x}), & s/z2/z1/y \\ p(z_1|x)p(s|z_1, \not{x})p(z_2|\not{s}, z_1, x)p(y|\not{s}, z_2, \not{x}, \not{x}), & s/z2/z1/y \\ p(s|x)p(z_1|s, x)p(z_2|\not{s}, z_1, x)p(y|\not{s}, z_2, \not{x}, \not{x}), & z1/s/z2/y \\ p(s|x)p(z_1|s, x)p(z_2|\not{s}, z_1, x)p(y|\not{s}, z_2, \not{x}, \not{x}), & s/z1/z2/y \\ p(s|x)p(z_1, z_2|s, x)p(y|\not{s}, z_2, \not{x}, \not{x}), & S-\text{Inference} \\ \dots \end{cases}$$

Z-first and S-first inference graph models relevant for generative models D–G, J, and K

Z

(b) Model Ds

(h) Model Gs

Y

X

S

(a) Model Dz

(g) Model Gz

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