Real-Time Optimization: A Memory-based Concurrent Extremum Seeking Approach

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Real-Time Optimization: A Memory-based Concurrent Extremum Seeking Approach

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Abstract: The increasing availability of information-rich data sets offers an invaluable opportunity to complement and improve the performance of existing model-based feedback algorithms. Following this principle, in this paper we present a novel type of extremum seeking controllers for static maps, which makes use of current and recorded data in order to solve a convex optimization problem characterized by a variational inequality. The optimization dynamics synergistically combine ideas from concurrent learning and classic neuro-adaptive extremum seeking in order to dispense with the assumption of requiring a persistence of excitation condition in the closed-loop system. Using Lyapunov-based tools and singular perturbation techniques, we show that, for a general class of optimization dynamics, it is possible to tune the parameters of the controller to guarantee convergence in finite time to a neighborhood of the set of optimizers. The results are illustrated in a scalar constrained minimization problem, as well as in a distributed resource allocation problem defined in a multi-agent system.

Keywords: Extremum seeking, adaptive control, optimization, multi-agent systems.

1. INTRODUCTION

Recent years have seen an increase of interest in data-driven control techniques that exploit the large amounts of data available in many fields. However, while areas such as machine learning, reinforcement learning, and gradient-free optimization have made significant breakthroughs during the last years, there is still a need to develop data-driven algorithms with provable and certifiable robustness properties for certain applications where safety is a critical concern. This necessity has motivated the study of novel robust feedback mechanisms that combine data-driven techniques with model-based principles, i.e., learning-based controllers, see for instance Benosman (2016), Vamvoudakis et al. (2015) and the recent survey paper of Poveda et al. (2018).

In the area of real-time optimization, extremum seeking (ES) control has emerged as a promising feedback-based methodology with certifiable stability and robustness properties. The literature of ES goes back to the early works of Leblanc (1922) and Blackman (1962). However, it was the seminal works of Krstić and Wang (2000) and Ariyur and Krstić (2003) that renewed the interest in this type of self-optimizing dynamics by providing a rigorous stability analysis based on averaging and singular-perturbation theory for ordinary differential equations (ODEs). Semi-global practical results were later developed in Tan et al. (2006), and further generalized in Nešić et al. (2010, 2012) for a broader class of optimization dynamics. Other averaging-based approaches have been recently studied in Dürr et al. (2013); Grushkovskaya et al. (2018); Mills and Krstić (2014); Feiling et al. (2018) and Poveda and Teel (2017a). ES architectures based on sampled-data systems have been studied in Popovic et al. (2006); Khong et al. (2013) and in Poveda and Teel (2017b). Stochastic approaches were developed in Liu and Krstic (2010); Poveda et al. (2015) and in Khong et al. (2015). In the context of ES with a persistence of excitation (PE) condition, the works of Guay and Zhang (2003); Guay et al. (2015); Dougherty and Guay (2017) and references therein, have developed significant results for single-agent and multi-agent optimization problems. Hybrid ES architectures that combine continuous-time and discrete-time dynamics, and that require a PE condition during the flows were also presented in Poveda et al. (2017). These techniques have all been developed under the assumption that the controller has access only to current data. However, as shown in Chowdhary and Johnson (2010), in some adaptive controllers it is possible to relax the classic PE condition by using recorded data that is “sufficiently rich”. Nevertheless, to the best of our knowledge, the development and analysis of ES architectures that use current
and past data concurrently during the seeking process are absent in the literature.

In this paper, we present a novel feedback architecture for memory-based extremum seeking control that dispenses with the classic PE condition. In particular, we show that the ideas behind concurrent learning and extremum seeking control can be synergistically combined to achieve real-time optimization in settings where there exists sufficiently rich stored data. The rest of this paper is organized as follows: In Section 2 we present some preliminaries and definitions. In Section 3 we present the main results. Section 4 presents a numerical example, and finally Section 5 follows: In Section 2 we present some preliminaries and definitions. In Section 3 we present the main results. Section 4 presents a numerical example, and finally Section 5 ends with the conclusions.

2. PRELIMINARIES AND NOTATION

The set of (nonnegative) real numbers is denoted by \((\mathbb{R}_{\geq 0})\). \(\mathbb{R}\). The set of (nonnegative) integers is denoted by \((\mathbb{Z}_{\geq 0})\). We use \(\mathbb{B}\) to denote a closed unit ball of appropriate dimension, \(p\in \mathbb{B}\) to denote a closed ball of radius \(p\), and \(X + \rho \mathbb{B}\) to denote the union of all sets obtained by taking a closed ball of radius \(\rho\) around each point in the set \(X\). We use \(\overline{X}\) to denote the closed convex hull of \(X\), \(\mathcal{X}\) to denote its closure, and \(\text{int}(\mathcal{X})\) to denote its interior. A constrained system of the form

\[ x = F(x), \quad x \in C, \quad (1) \]

is said to render a compact set \(\mathcal{A}\) uniformly globally asymptotically stable if there exists a \(K\mathcal{L}\) function \(\beta\) such that \(\|x(t)\| \leq \beta(|x(0)|, t)\), for all \(t \in \text{dom}(x)\) and all \(x(0) \in C\). If \(\text{dom}(x) = [0, \infty)\), the solution \(x\) is said to be complete. If the flow set \(C\) is forward invariant, then all solutions of the constrained system (1) are complete. We say that \(f \in C^2\) if \(f\) is at least twice continuously differentiable.

3. EXTREMUM SEEKING WITH MEMORY

The standard extremum seeking setting in static maps aims to solve an optimization problem of the form

minimize \( f(x) \) \quad (2a)

subject to \( x \in K\), \quad (2b)

where \( f : \mathbb{R}^n \to \mathbb{R}\) is an unknown continuously differentiable objective function defined on an open set containing the closed set \(K \subset \mathbb{R}^n\). The objective function is assumed to be measurable only by measurements or evaluations. If \(f\) is also a convex function and the set \(K\) is convex, a point \(x^* \in K\) is a solution of (2) if and only if it satisfies the variational inequality

\((x - x^*)^T \nabla f(x^*) \geq 0, \quad \forall x \in K, \quad (3)\)

where \(\nabla f(x)\) is the gradient of \(f(x)\), (Rockafellar and Wets, 1998, Thm 6.12). Consequently, for convex optimization problems, ES control can be seen as a feedback-based data-driven algorithm designed to converge to the solutions of (3). Since, in general, it is difficult to verify a priori a convexity condition on the cost function \(f(x)\), typical ES controllers assume that (3) holds only “locally”.

Because of this, and in order to impose some regularity conditions to problem (2), we make the following standard assumption.

Assumption 1. The function \(f\) is convex and continuously differentiable on an open set \(D \supset K\), and the set \(K\) is compact, convex, and nonempty.

We denote by \(\mathcal{A}\) the set of all feasible points \(x^*\) satisfying (3), that is

\[ \mathcal{A} := \{x^* \in K : (x - x^*)^T \nabla f(x^*) \geq 0, \quad \forall x \in K\}. \quad (4) \]

By Assumption 1, this set is nonempty and compact (Facchinei and Pang, 2003, Corollary 2.2.5).

There exist several approaches in the literature to solve the ES problem (2), including averaging-based architectures, sampled-data based controllers, stochastic algorithms, and neuro-adaptive schemes, to just name a few. In this paper, we focus on the later type of ES controllers, which exploit ideas from adaptive control and neural networks in order to obtain uniform approximations of \(\nabla f(x)\) on compact sets.

3.1 Feedback Structure

Consider an ES controller characterized by the dynamics

\[ \dot{w} = F_w(w, \phi(x), e), \quad (5a) \]

\[ \dot{e} = e_F(w^T \phi(x) + z), \quad z \in C_z \quad (5b) \]

where \(z := [x^T, e^T]^T \in \mathbb{R}^{n+2}\) and \(s \in \mathbb{R}\) is an auxiliary state of dimension \(s \in \mathbb{R}\). The specific forms of the mapping \(F_w\) and the set \(C_z \subset \mathbb{R}^{n+2}\) are application-dependent and designed to solve problem (3). The error signal \(e\) in (5a) is defined as

\[ e = w^T \phi(x) - f(x), \quad (6) \]

and the state \(\hat{w} \in \mathbb{R}^p\) is an auxiliary state used to estimate the ideal weights \(w^* \in \mathbb{R}^n\) of a parameterization of the objective function, given by

\[ f(x) = w^T \phi(x) + e(x), \quad \forall x \in K, \quad (7) \]

where \(w^* \in \mathbb{R}^p\) belongs to the set of ideal weights

\[ \mathcal{W}_{\phi,K} := \{w^* \in \mathbb{R}^p : |f(x) - w^{*T} \phi(x)| = e(x), \quad x \in K\}, \]

which we assume to be compact, i.e., there exists \(\bar{w} > 0\) such that \(\mathcal{W}_{\phi,K} \subset \bar{w} \mathbb{B}\). The vector valued regressor function \(\phi : K \to \mathbb{R}^p\) is assumed to be known and \(\phi \in C^2\). The regressor function \(\phi := \{\phi_1, \phi_2, \ldots, \phi_p\}\) should be selected such that the functions \(\phi_j, j \in \{1, \ldots, p\}\), define a complete independent basis set for \(f(x)\). Typical choices of \(\phi_j\) include quadratic functions, radial basis functions, or sigmoid functions, see Vanmoudakis and Lewis (2010).

We shall need the following technical assumption on the approximation (7).

Assumption 2. The approximation error function \(e(x)\) in (7) is continuously differentiable.

By calculating the gradient of (7) we can obtain the following parameterized expression for \(\nabla f\):

\[ \nabla f(x) = w^* \nabla \phi(x) + \nabla e(x), \quad \forall x \in K, \quad (8) \]

where \(\nabla \phi(x)\) is the Jacobian of \(\phi(x)\). Since \(K\) is compact and the mappings \(\nabla f(x)\) and \(\nabla e(x)\) are continuous, by the Weierstrass high-order approximation theorem (Dudley, 2002, Thm. 2.4.11), the approximation error converges to zero as the number of basis \(p\) increases, i.e., \(e(x) \to 0\) and \(\nabla e(x) \to 0\) as \(p \to \infty\), uniformly on \(K\). Moreover, due to Assumption 2, the compactness of \(K\) and the fact that \(\phi(x) \in C^2\), we have that \(\phi(x), \nabla \phi(x), e(x), \text{ and } \nabla e(x)\) are all uniformly bounded in \(K\).

Traditionally, the parametrization (8) allows to design learning dynamics (5a) with solutions converging to the set
\[ W_{x,k}, \text{ provided the time-varying regressor vectors } \phi(x(t)) \text{ satisfy a persistence of excitation condition.} \]

### 3.2 Online Learning with Persistance of Excitation

Let \( \tilde{f}(x) \) be an approximation of the objective function (7), defined as

\[ \tilde{f}(x) := \tilde{w}^\top \phi(x), \quad \forall \ x \in K. \quad (9) \]

Let \( \tilde{w} = \tilde{w} - w^* \). Then, using (6), (7) and (9) we have that

\[
e(t) = \tilde{f}(x(t)) - f(x(t)) = \tilde{w}^\top \phi(x(t)) - e(x(t)).
\]

Consider the mapping (5a), defined as

\[ F_w(\tilde{w}, \phi(x), e) := -\alpha \frac{\phi(x(t))}{(1 + \phi(x(t))^\top \phi(x(t)))^2} e(t), \quad (10) \]

where \( \alpha > 0 \) is a tunable constant. If \( e(x) = 0 \), it is well-known, e.g., Ioannou and Sun (2012), Vamvoudakis and Lewis (2010), Narendra and Annaswamy (1987), that all solutions of (5a) with vector field (10) will satisfy \( \tilde{w}(t) \to w^* \) as \( t \to \infty \), uniformly in \( (t_0, \tilde{w}(0)) \), if and only if the normalized regressor function \( \phi(x(t)) \), defined as

\[ \tilde{\phi}(x(t)) := \frac{\phi(x(t))}{1 + \phi(x(t))^\top \phi(x(t))}, \]

is persistently exciting, i.e., there exists a \( T > 0 \) and \( \gamma > 0 \) such that

\[ \int_{t}^{t+T} \tilde{\phi}(\tau) \tilde{\phi}(\tau)^\top d\tau \geq \gamma I, \quad (12) \]

for all \( t \geq t_0 \) and all \( t_0 > 0 \). When \( e(x) \neq 0 \) in (10), the PE condition (12) guarantees convergence of \( \tilde{w}(t) \) to a neighborhood of \( w^* \) that is proportional to the upper bound of \( e(x) \), see for instance Vamvoudakis and Lewis (2010).

Since it needs to be satisfied for all present and future time, the PE condition (12) can be restrictive and unfeasible for certain applications. On the other hand, as shown in Chowdhary and Johnson (2010), if the learning dynamics use current data \( \phi(x(t)) \) then the PE is not required, which is complemented with a finite sequence of stored past data \( \{\phi(x(t_k))\}_{k=1}^{\tilde{w}} \), learning of the optimal weights \( w^* \) can still be achieved provided the stored data is sufficiently rich.

### 3.3 Online Learning with Memory

To establish the main ideas behind memory-based ES control, let \( k \in \{1, 2, \ldots, \tilde{k}\} \) denote the index of a stored data point \( x_k \), i.e., \( x_k = x(t_k) \), and let \( \phi(x_k) \) be the regressor vector evaluated at that point. We still denote by \( e(t) \) in (10) the estimation error corresponding to the data collected at the current time \( t \), but we now also introduce an estimation error associated to the data previously collected at time \( t_k \), given by

\[
e(t_k, t) = \tilde{f}(x(t_k)) - f(x(t)), \quad (13a)
\]

\[
= \tilde{w}(t)^\top \phi(x_k) - e(x_k). \quad (13b)
\]

for all \( t_k \in \{1, 2, \ldots, \tilde{k}\} \). Note that the estimation error \( \tilde{w} \) still depends on the current time \( t \).

**Definition 3.** The sequence of stored data \( \{\phi(x_k)\}_{k=1}^{\tilde{k}} \) is said to be \( k \)-sufficiently rich if the following inequality is satisfied

\[ \sum_{k=1}^{\tilde{k}} \tilde{\phi}(x_k) \tilde{\phi}(x_k)^\top > 0. \quad (14) \]

with \( \tilde{\phi}(x_k) := \frac{\phi(x(t_k))}{1 + \phi(x(t_k))^\top \phi(x(t_k))}. \)

According to definition (3), the \( k \)-sequence of stored data is \( k \)-sufficiently rich if its elements are sufficiently different to form a basis for the parameterized uncertainty during the window of discrete time \( \{1, 2, \ldots, \tilde{k}\} \). Indeed, by defining the regressor matrix

\[ \phi^{\text{mem}} := [\phi(x_1), \phi(x_2), \ldots, \phi(x_k)]^\top, \quad (15) \]

the condition \( \text{rank}(\phi^{\text{mem}}) = p \) is sufficient to satisfy (14). Using data that satisfies (14), we can consider learning dynamics (5a) that dispense with the PE condition. To streamline the presentation of the algorithm, and with some abuse of notation, we will use \( t_0 = t \) to denote the current time, and we define

\[ e(t_0, t) := e(t). \quad (16) \]

Using this notation, we replace the mapping \( F_w \) in (10) by the data-driven mapping

\[ F_w(\tilde{w}, \phi(x), e) := -\alpha \sum_{k=0}^{\tilde{k}} \frac{\phi(x(t_k))}{[\phi(x(t_k))^\top \phi(x(t_k))] + 1} e(t_k, t). \quad (17) \]

**Lemma 4.** Suppose that Assumptions 1-2 hold and that the signals \( \phi(x(t_k)) \) in (17) are \( k \)-sufficiently rich. Then, for each pair \( \nu, c \in \mathbb{R}_{>0} \) such that \( \nu < \sqrt{2c} \) there exists a sufficiently large \( p^* \in \mathbb{Z}_{>0} \) such that for each \( p > p^* \) there exists a UGAS compact set \( \mathcal{A}_p \subset \mathbb{M}_{\phi,C} + \nu \mathbb{B} \) for the dynamics (5a) restricted to the positive invariant compact set \( \mathbb{M}_{\phi,C} + \nu \mathbb{B} \). Moreover, the rate of convergence is exponential outside the set \( \mathbb{M}_{\phi,C} + \nu \mathbb{B} \).

**Proof:** The proof of the following Lemma follows similar ideas as the proofs in Chowdhary and Johnson (2010) and Vamvoudakis and Lewis (2010). We divide the proof in three main steps.

**Step 1:** Let \( \tilde{w} := \tilde{w} - w^* \), and consider the error dynamics

\[
\dot{\tilde{w}} = -\alpha \sum_{k=0}^{\tilde{k}} \phi(x(t_k)) \tilde{\phi}(x(t_k))^\top \tilde{w}
+ \alpha \sum_{k=0}^{\tilde{k}} \tilde{\phi}(x(t_k))^\top \phi(x(t_k)) + 1 e(x_k), \quad (18)
\]

which can be written as

\[
\dot{\tilde{w}} = -\alpha \tilde{\phi}(x(t)) \tilde{\phi}(x(t))^\top \tilde{w} - \alpha \sum_{k=1}^{\tilde{k}} \tilde{\phi}(x(t_k)) \tilde{\phi}(x(t_k))^\top \tilde{w}
+ \alpha \sum_{k=1}^{\tilde{k}} \tilde{\phi}(x(t_k))^\top \phi(x(t_k)) + 1 e(x_k).
\]

Define \( P(t) \) as

\[ P(t) := \tilde{\phi}(x(t))^\top \phi(x(t)) + \sum_{k=1}^{\tilde{k}} \tilde{\phi}(x(t_k))^\top \phi(x(t_k)). \quad (20) \]

Using that \( \{\tilde{\phi}(x(t_k))\}_{k=1}^{\tilde{k}} \) is \( k \)-sufficiently rich, that \( x(t_k) \) is constrained to a compact set, and that \( \phi(\cdot) \) is continuous, there exists \( \delta_1, \delta_2 > 0 \) such that
\[ \delta_2 I_p \geq \sum_{k=1}^{\bar{k}} \tilde{\phi}_k(x(t_k)) \tilde{\phi}_k(x(t_k))^\top \geq \delta_1 I_p, \]  

and since \( \tilde{\phi}(x(t)) \tilde{\phi}(x(t))^\top \) is symmetric, positive semidefinite, and uniformly bounded, there exists \( \delta_3 > 0 \) such that

\[ \delta_3 I_p \geq P(t) \geq \delta_1 I_p, \]

for all \( t \geq t_0 \) and all \( t_0 \geq 0 \).

**Step 2:** Let \( \rho(x) \) be given by

\[ \rho(x) = \alpha \sum_{k=0}^{\bar{k}} \frac{\tilde{\phi}(x(t_k))}{|\tilde{\phi}(x(t_k))|} \phi(x(t_k)) + 1 \]  

where we used the upper bound in (22) and the definition of \( P \) in (20). If the approximation error \( \epsilon(x) \) is zero in (7), we have that \( \rho(x) = 0 \) and \( \bar{w}(t) \) converges exponentially fast to zero. Moreover, the level sets \( L_c := \{ \bar{w} \in \mathbb{R}^p : V(\bar{w}) \leq c \} \) are positive invariant for each \( c > 0 \). On the other hand, when \( \epsilon(x) \neq 0 \), by the definition of the entries \( \rho \) in (23), the fact that \( x \) is constrained to a compact set, the continuity of \( \phi(\cdot) \), and the approximation properties of the regressions in (7), for any \( \nu > 0 \) there exists a sufficiently large \( p^* \in \mathbb{Z}_{>0} \) such that for all \( |p| \geq p^* \) the residual term satisfies \( |\rho(x)| < \nu \). Thus, for \( |p| \geq p^* \) equation (25) satisfies

\[ \dot{V} \leq \delta_{\delta_3} |\bar{w}|^2 + \nu |\bar{w}|, \]

where \( \nu := (\delta_{\delta_3})^{-1} \nu \). Combining inequalities (24) and (26) we get ultimate boundedness of \( W(t) \) with residual set proportional to \( \nu \).

**Step 3:** Finally, since Step 2 implies that for each \( \bar{w}(0) \in L_c \) and each \( \nu > 0 \) there exists a sufficiently large \( p^* > 0 \) such that for each \( |p| \geq p^* \) there exists a \( T > 0 \) such that \( \bar{w}(t) \in vB \) for all \( t \geq T \), by (Goebel et al., 2012, Corollary 7.7), there exists an asymptotically stable set \( A_p \subset vB \) for the error dynamics (18). Restricting this dynamics to evolve in the set \( L_c \), which is positive invariant, we get \( uGAS \) of \( A_p \) for the error dynamics with flow set \( C = L_c \).

Having established a data-driven learning mechanism to estimate the gradient \( \nabla f \), we can proceed to design the optimization dynamics to solve problem (3).

### 3.4 Robust Gradient-Based Optimization Dynamics

The dynamics (5b) are designed under the ideal assumption that \( \nabla f = \bar{w}^\top \nabla \phi(x) \). In particular, to solve the VI problem (3) we consider gradient systems of the form

\[ \dot{z} = F_x(\nabla f(z), z), \quad z \in C_z, \]  

where the mapping \( F_x \) and the set \( C_z \) are designed based on the following Assumption.

**Assumption 5.** The dynamics (27) satisfy the following:

(a) The mapping \( F_x \) is continuous with respect to both arguments.

(b) The set \( C_z \) satisfies \( C_z := K \times S \), where \( S \subset \mathbb{R}^r \) is a compact set.

(c) There exists a nonempty compact set \( S \subset \mathbb{R}^r \) such that the set \( A \times S \) is \( uGAS \).

(d) There exists an \( \delta > 0 \) such that for each measurable function \( c : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n \) satisfying \( sup_{t \geq 0} |c(t)| \leq \delta \), the perturbed system

\[ \dot{z} = F_x(\nabla f(z) + c, z), \quad z \in C_z, \]

generates complete solutions from each \( z(0) \in C_z \). □

In words, Assumption 5 asks that system (27) solves problem (2) under the assumption of perfect knowledge of the gradient, and generates complete solutions under vanishing perturbations acting on the gradient. In some cases, item (d) can be relaxed and complete solutions are only required from compact subsets of \( C_z \). Examples of dynamics satisfying Assumption 5 will be presented in Section 4.

### 3.5 Main Result

Having characterized the memory-based learning dynamics (17) and the gradient dynamics (27), we are ready to present the main result of this paper.

**Theorem 6.** (ES with Memory) Suppose that Assumptions 1, 2 and 6 hold. Consider the dynamics (5) with \( F_x \) given by (17) and \( (F_x, C_z) \) given by (27). Suppose that the sequence of data \( \{\phi(x(t_k))\}_{k=1}^{\bar{k}} \) in (17) is \( k \)-sufficiently reach. Then, for each \( \nu > 0 \) there exists a \( p^* \in \mathbb{Z}_{>1} \) such that for \( |p| > p^* \) there exists \( \epsilon^* \in \mathbb{R}_{>0} \) such that for each \( \epsilon \in (0, \epsilon^*) \) there exists a \( T_{\nu, \epsilon} \) such that

\[ x(t) = \bar{x} + \nu B, \]

for all \( t \geq T_{\nu, \epsilon} \).

**Proof:** The proof uses tools from singular perturbation theory, robustness results for well-posed systems, and \( \Omega \)-limit sets. In particular, the closed-loop system is given by

\[ \dot{\bar{w}} = -\alpha \sum_{k=0}^{\bar{k}} \frac{\phi(x(t_k))}{|\phi(x(t_k))|} \phi(x(t_k)) + 1 \]  

\[ \dot{z} = \epsilon F_x(\bar{w}^\top \nabla \phi(x_k), z), \quad z \in C_z. \]

Let \( \tau = \epsilon t \) be a new time scale. For \( p \) sufficiently large, we will analyze system (30) as a perturbed version of the nominal system in the \( \tau \)-time scale, given by

\[ \dot{\bar{w}} = -\alpha \sum_{k=0}^{\bar{k}} \frac{\phi(x(t_k))}{|\phi(x(t_k))|} \phi(x(t_k)) + 1 \]  

\[ \dot{z} = F_x(\bar{w}^\top \nabla \phi(x_k), z), \quad z \in C_z, \]

where the approximation error \( \epsilon(x) \) was set to zero.

For \( \epsilon > 0 \) sufficiently small, system (31) is in singular-perturbation form, see Sanfelice and Teel (2011), Wang et al. (2012), Teel et al. (2003). The boundary layer dynamics are given by (17), which, by Lemma 4, guarantee the existence of a \( p \) sufficiently large such that
converge exponentially fast to \( w^* \in \mathcal{W}_{\rho,K} \) thanks to the sufficiently reach condition. Therefore, the reduced system corresponds precisely to the gradient dynamics (27), which, by item (c) in Assumption 5, renders UGAS the set \( \mathcal{A} := \mathcal{A} \times \mathcal{S} \). Applying (Wang et al., 2012, Thm. 1) for purely continuous-time systems we obtain that for each \( \nu/2 > 0 \) there exists \( \varepsilon^* > 0 \) such that for each \( \varepsilon \in (0, \varepsilon^*) \) there exists a \( T_{\nu} \) such that \( z(\tau) \in \mathcal{A} + 0.5\nu \mathbf{B} \) for all \( t \geq T \). In turn, by completeness of solutions, this implies the existence of a UGAS set \( \mathcal{A}_x \subset \mathcal{A} + 0.5\nu \mathbf{B} \times L_x \). Let 
\[
F_y(y) := [F_x^T, F_z^T]^T, \quad y = [\hat{w}^T, z^T]^T, \quad \text{and} \quad C_L = L_x \times C_z.
\]
By continuity of \( \nabla \phi(x) \) for each \( \varepsilon > 0 \) there exists \( \rho > 0 \) such that \( F(y + \mathbf{B}) + \varepsilon \mathbf{B} \subset \overline{\text{conv}} F(y + \rho \mathbf{B}) \cap C + \rho \mathbf{B} =: F_r(y) \). Since the right-hand side of (31) is continuous, and the sets \( C_L \) and \( L_x \) are closed, the UGAS properties of \( \mathcal{A}_x \) are maintained by the set \( \frac{d\hat{w}}{dt} \in F_r(y), \ y \in C_\rho := \{ y : (y + \rho \mathbf{B}) \cap C \neq \emptyset \} \) for \( \rho > 0 \) sufficiently small and in a semi-global practical way (Goebl et al., 2012, Thm. 7.21). Since \( \rho > 0 \) can be decreased by decreasing \( \varepsilon \), there exists \( \rho^* \) such that for all \( [\rho] \geq \rho^* \), the approximation error \( e(x) \) satisfies \( |e(x)| \leq \varepsilon \) sufficiently small and there exists \( T_{\rho} > 0 \) such that every solution of \( \frac{d\hat{w}}{dt} \in F_r(y), \ y \in C_\rho \) satisfies \( y(\tau) \in \mathcal{A}_x + \nu/2 \mathbf{B} \subset \mathcal{A} + \nu \mathbf{B} \times L_x \). Given that solutions of the perturbed system \( \frac{d\hat{w}}{dt} \in F_r(y), \ y \in C_\rho \) are also solutions of system (30) in the \( \tau \)-time scale, the solutions of (30) satisfy \( y(t) \in \mathcal{A} + \nu \mathbf{B} \times L_x \) for all \( t \geq T_{\rho}/\varepsilon \). This establishes inequality (29).

4. APPLICATION: EXTREMUM SEEKING WITH MEMORY AND PROJECTION

In this section we present two applications of the family of memory-based ES algorithms described by the dynamics (5). We consider a ES problem (2) where the cost function \( f(x) \) satisfies Assumption 1, and the following memory-based ES dynamics are used
\[
\begin{align}
\dot{w} &= -\sum_{k=0}^{K} \frac{\phi(x(t_k))}{\phi(x(t_k)) + 1} \varepsilon(t_k), \\
\dot{x} &= \varepsilon [-x + P_K(x - \hat{w}^T \nabla \phi(x))],
\end{align}
\tag{32}
\]
where \( P_K(x) = \arg\min_{u \in K} \|x - u\|_2 \). The projection dynamics (32b) are Lipschtiz continuous since \( P_K(x) \) satisfies the non-expansive property \( \|P_K(x) - P_K(y)\| \leq |x - y| \). Also, dynamics (32b) render forward invariant the set \( K \) (Xia and Wang, 2000, Thm. 3.2). Moreover, when \( \hat{w}^T \nabla \phi(x) = \nabla f(x) \) small perturbations acting on the gradient do not affect the forward invariance of \( K \) since \( P_K(x - \nabla f(x) + \varepsilon) \subset K \) for any \( \varepsilon \). Additionally, since by Assumption 1 the function \( f \) is convex, we have that \( \nabla f(x) \) is a monotone gradient mapping (Rockafellar and Wets, 1998, Thm. 12.17). Then, by (Gao, 2003, Thm. 3), the monotonicity and Lipschitz continuity of \( \nabla f(x) \), plus the convexity and closedness of \( K \), imply the convergence of the solutions of the projected gradient dynamics \( \dot{x} = \varepsilon [-x + P_K(x - \nabla f(x))] \) to the set \( \mathcal{A} \). Since \( K \) and \( \mathcal{A} \) are compact, and the vector field in (32b) is Lipschitz continuous, the convergence is uniform in \( K \). Therefore, Assumption (5) is satisfied.

Figures 1 and 2 show simulations of the solutions of the dynamics (32) with and without memory for the case when

\begin{align*}
\text{Fig. 1. Evolution in time of the state } x(t) \text{ with PE condition (12) on the regressor vector } \phi(t). \text{ The dotted lines describe the limits of the set } K. \\
\text{Fig. 2. Evolution in time of the state } x(t) \text{ with no PE condition and using memory-based ES with recorded sequence } \{ \phi(x_k) \} .
\end{align*}

\[
f(x) = (x - 2)^2, \quad K = [1, 4].
\]

For the simulation shown in Figure 1 no recorded data was used and a dithering signal was injected into the closed-loop system to satisfy the standard PE condition (12). On the other hand, in Figure 2 we show two simulations where memory-based ES with recorded data was used. The algorithms uses a vector-valued regressor \( \phi : \mathbb{R} \rightarrow \mathbb{R}^4 \), given by \( \phi(x) = [x^2, x, 1]^T \). For the case when \( \text{rank}(\phi^{\text{mem}}) = 3 \) condition (14) is satisfied and the memory-based ES converges to the optimizer of the cost function. On the other hand, when \( \text{rank}(\phi^{\text{mem}}) = 1 \) the inset shows that, as expected, the algorithm does not converge to \( x^* \). The recorded data \( \{\phi(x_k)\}_k \) was generated by exciting the state during 5 seconds and sampling the state every 0.1 seconds. The red dotted lines describe the limits of the set \( K \).

5. CONCLUSION

We presented a novel class of memory-based ES controllers that exploits ideas from concurrent learning to exploit information-rich data sets and to avoid the injection of persistent dithering signals in the closed-loop system. The proposed scheme uses recorded data during the learning phase concurrently with current data in order to guarantee convergence to an \( \varepsilon \)-neighborhood of the solution of the extremum seeking problem. Our results are general enough to be applied to different types of optimization dynamics that evolve on compact sets and solve general classes of convex variational inequalities, including Lipschitz continuous projected gradient descent. We anticipate that our results can be extended to settings where the cost function is generated by a stable dynamical system, as well as to optimization problems with slowly varying cost functions.

REFERENCES
