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On Mean Field Games for Agents with Langevin Dynamics

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***Abstract*—Mean Field Games (MFG) have emerged as a viable tool in the analysis of large-scale self-organizing networked systems. In particular, MFGs provide a game-theoretic optimal control interpretation of the emergent behavior of non-cooperative agents. The purpose of this paper is to study MFG models in which individual agents obey multidimensional nonlinear Langevin dynamics, and analyze the closed-loop stability of fixed points of the corresponding coupled forward-backward PDE systems. In such MFG models, the detailed balance property of the reversible diffusions underlies the perturbation dynamics of the forward-backward system. We use our approach to analyze closed-loop stability of two specific models. Explicit control design constraints which guarantee stability are obtained for a population distribution model and a mean consensus model. We also show that static state feedback using the steady state controller can be employed to locally stabilize a MFG equilibrium.**

I. Introduction

Large scale non-cooperative multi-agent systems involving coupled costs were introduced as mean field games (MFG) by Huang et. al [1] and Lasry et. al [2]. Key ideas in this theory are the rational expectations hypothesis, infinitely many anonymous agents and that individual decisions are based on statistical information about the collection of agents. Subsequently, this theory has become a viable tool in the analysis of large-scale, self-organizing networked-systems, and provides a game-theoretic optimal control interpretation of the the notion of *emergent behaviour* in the non-cooperative setting. In the continuum approach, MFG models are synthesized as standard [3] stochastic optimal control problems (OCP). Fully coupled Fokker Planck (FP) and Hamilton Jacobi Bellman (HJB) equations governing agent density and value functions constitute the mean field (MF) optimality system. MFG models have been constructed to study several naturally occurring and engineered large-scale networked systems, including traffic [4], financial [5], energy [6], and biological systems [7].

A characteristic feature of MFGs is the ability to model interaction between networked agents by designing a suitable cost function. If the cost function has only local density dependence and is strictly increasing, steady state solutions to the MF system are unique [8] in several cases. In the absence of monotonicity, MFGs exhibit non-unique solutions and related *phase transitions* [7], [9], [10]. Since real-world large-scale networked systems often possess several 'operating regimes', non-monotonicity in the corresponding MFG models is expected to be the norm, rather than an exception. Closed-loop stability analysis of MFG models that do not satisfy the monotonicity condition has to be done on a case-by-case basis. A given fixed point of the MFG is called (linearly) closed-loop stable if any perturbation to the fixed-point density decays to zero under the action of the control, where both the density and control evolution are computed using the (linearized) coupled forward-backward system of FP-HJB PDEs.

Guéant [11] studied the stability of an MFG model with a negative log density cost. Stability of MFGs with nonlocal cost coupling was considered for a Kuramoto oscillator model by Yin et. al [9] and a mean consensus cost by Nourian et. al [12], [13]. A common limitation of these prior works is that the agents dynamics are assumed to be simple integrator systems. The MF approach to large-scale systems with nonlinear agent dynamics has been used to model crowds [14], Brownian particles in non-equilibrium thermodynamics [15], and robotic systems [16]. In particular, overdamped Langevin systems that we consider in this paper have been used in MF formulations of deep neural networks [17], [18], and flocking with self-propulsion [19]. Certain multi-agent decision making problems [20], [21] can also be studied within this framework. In our recent work [7], we analytically and numerically explored phase transitions in MFG models consisting of agents with nonlinear passive dynamics.

We expand upon the idea introduced in [7], and present rigorous closed-loop linear stability analysis for quadratic MFG models [10] with dynamics of individual agents lying in the general class of controlled *reversible diffusions*. An example of such diffusions are the overdamped Langevin (simply Langevin for brevity) dynamics given in (1), while the simplest case is that of integrator systems. The key idea is that the *detailed balance* property of the generator of controlled reversible diffusions, and the resulting spectral properties of the linearized MFG system, allow for generalization of existing stability analysis techniques to this larger class of MFG systems. Furthermore, we demonstrate that static state feedback using the steady state controller can be employed to (sub-optimally) locally stabilize a MFG equilibrium.

In section II, we describe the class of MFG models treated in this paper. In section III, we present the arguments detailing the main ideas for stability analysis for this class of models. Detailed analysis of closed-loop linear stability of steady states for (i) a population model with *local* cost coupling and (ii) consensus model with *nonlocal* cost coupling are presented next, which illustrate the key ideas in our approach. The population model consists of a general class of nonlinear controlled

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Langevin agent dynamics with a negative log density cost [11]. In section IV we present technical conditions required for stability on the stationary solution and control parameters, and local stability results for this model. This analysis generalizes the stability analysis for the integrator dynamics case presented in [11]. The stability analysis does not require explicit analytic solution of the stationary MF optimality system and the related eigenbasis, which is in contrast to prior works that exploit quadratic-Gaussian solutions and associated Hermite basis. Thus, as in standard equilibrium theory, the presented techniques can be potentially used to analyze the dynamics of distributions in relation to various stationary points resulting from MFG optimality. The consensus model has flocking cost as in [13]. In section V we present stationary solutions, control design parameter constraints and linear stability results for this model in which agents obey Langevin dynamics with quadratic potential. Our results on this model generalize those of [13] concerned with integrator agent dynamics.

Finally, in section VI, the action of the MF steady state controller on a population of agents in a MFG with nonlinear Langevin dynamics is considered. We show that a population of agents with perturbed (non Gaussian) initial densities will decay to the (closest) stationary density under the action of static feedback given by the corresponding steady state controller.

II. Mean Field Game Model

In this section, we first introduce some notation and then describe the MFG model treated in this paper. Vector inner products are denoted by $a{\cdot}b$, the induced Euclidean norm by $|a|$ and its square by $a^2 := |a|^2$. ∂_t denotes partial derivative with respect to t while ∇, $\nabla\cdot$ and Δ denote the gradient, divergence and Laplacian operations respectively. $L^2(g\ \mathrm{d}x;\mathbb{R}^d)$ denotes the class of g-weighted square integrable functions of $\mathbb{R}^d$. The norm of a function f and inner product of functions f_1, f_2 in this class is denoted by $||f||_{L^2(g\ \mathrm{d}x;\mathbb{R}^d)}$ and $\langle f_1, f_2\rangle_{L^2(g\ \mathrm{d}x;\mathbb{R}^d)}$ respectively.

Let $x_s,\ u(s) \in \mathbb{R}^d$ denote the state and control inputs of a representative agent which obeys controlled Langevin dynamics in the overdamped case, given by

$$\mathrm{d}x_s = -\nabla\nu(x_s)\mathrm{d}s + u(s)\mathrm{d}s + \sigma \mathrm{d}w_s \tag{1}$$

for every $s \geq 0$, driven by standard $\mathbb{R}^d$ Brownian motion, with noise intensity $0 < \sigma$ on the filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$. The smooth function $\nu : \mathbb{R}^d \to \mathbb{R}$ is called the Langevin potential and the control $u \in \mathcal{U} := \mathcal{U}[t,T]$, where $\mathcal{U}$ is the class of admissible controls [22] containing functions $u : [t,T] \times \mathbb{R}^d \to \mathbb{R}^d$. The MFG models treated in this work can be written as the following control problem subject to (1), with $t \geq 0$

$$\min_{u\in\mathcal{U}} J(u) := \mathbb{E}\left[\int_t^T \mathrm{e}^{-\rho s}\left(q(x_s, \mathrm{p}(s,\cdot)) + \frac{R}{2}u^2(s)\right)\mathrm{d}s\right], \tag{2}$$

where we denote the probability density of x_s by $\mathrm{p}(s,x)$ for every $s \geq 0$, with initial density being $x_t \sim \mathrm{p}(t,x)$, $q : \mathbb{R}^d \times L^1(\mathbb{R}^d) \to \mathbb{R}$ is a known deterministic function which has at most quadratic growth in (x, p) and $R > 0$ is the control cost. We assume that the functions in the class $\mathcal{U}$ and $\nabla\nu(x), q(x,\mathrm{p})$ are measurable. The value function is defined as $v(t,x) := \min_{u\in\mathcal{U}} J(u)$ given $x_t = x$. It can be seen by standard application of *dynamic programming* [23] as in ([2], [3]), that this control problem is equivalent to the following PDE system

$$-\partial_t v = q - \rho v - \frac{(\nabla v)^2}{2R} - \nabla v \cdot \nabla \nu + \frac{\sigma^2}{2}\Delta v \tag{3}$$

$$\partial_t \mathrm{p} = \nabla\cdot\left((\nabla\nu + \frac{\nabla v}{R})\mathrm{p}\right) + \frac{\sigma^2}{2}\Delta \mathrm{p} \tag{4}$$

with the optimal control $u^*(t,x) = -\nabla v/R$, the mass conservation constraint $\int \mathrm{p}(s,x)\mathrm{d}x = 1$ for all $s \geq 0$ and boundary constraints $\lim_{|x|\to+\infty} \mathrm{p}(t,x) = 0$, $\lim_{s\to+\infty} \mathrm{e}^{-\rho s} v(s,x_s) = 0$. These fully coupled equations identified as the HJB and FP PDEs comprise the MF optimality system. An infinite time horizon, that is $T \to +\infty$, leads to the stationary system

$$0 = q(x,\mathrm{p}^\infty) - \rho v^\infty - \frac{(\nabla v^\infty)^2}{2R} - \nabla v^\infty\cdot\nabla\nu + \frac{\sigma^2}{2}\Delta v^\infty, \tag{5}$$

$$0 = \nabla\cdot\left((\nabla\nu + \frac{\nabla v^\infty}{R})\mathrm{p}^\infty\right) + \frac{\sigma^2}{2}\Delta \mathrm{p}^\infty, \tag{6}$$

governing the fixed point pair $(v^\infty(x), \mathrm{p}^\infty(x))$ of steady state value and density functions, with constraints $\int \mathrm{p}^\infty(x)\mathrm{d}x = 1$, and $\lim_{s\to+\infty} \mathrm{e}^{-\rho s} v^\infty(x_s) = 0$. The optimal control is $u^\infty(x) = -\nabla v^\infty/R$.

Remark 1. *If the MFG model has a long-time-average (LTA) utility [9],*

$$\min_{u\in\mathcal{U}} J(u) := \lim_{T\to+\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T q(x_s, \mathrm{p}(s,\cdot)) + \frac{R}{2}u^2(s)\ \mathrm{d}s\right], \tag{7}$$

instead of the discounted version in (2)*, then the corresponding stationary optimality system consists of (*(5), (6)*), on observing the limit $\rho v^\infty \to \lambda$ in* (5) *as $\rho \to 0$, where λ is the optimal cost. Please see [24] and references therein for proof of this connection between the utilities. In this case, the time dependent, relative value function [25] obeys* (3) *wherein ρv is replaced by λ. Similarly, the perturbation system is obtained from* (17) *by setting $\rho = 0$. Thus, all the results in sections III, IV, V, VI can be directly extended to the LTA utility case.*

III. Perturbation System

The FP equation governing the density of an overdamped Langevin system is called the Smoluchowski PDE. From the form of the FP PDE (6), it can be interpreted as the Smoluchowski PDE for such a Langevin system with the restoring potential $\nu + v^\infty/R$. This interpretation allows us to obtain the analytical solution to the FP PDE as a Gibbs distribution, if the fixed point pair $(v^\infty, \mathrm{p}^\infty)$ of the MFG (5, 6) and the Langevin potential ν satisfy certain conditions. We denote $w(x) := \nu(x) + \frac{v^\infty(x)}{R}$ henceforth in this paper.

Lemma III.1. *If $v^\infty(x)$, $\nu(x)$ are smooth functions satisfying $\lim_{|x|\to+\infty} w(x)=+\infty$ and $\exp\left(-\frac{2}{\sigma^2}w(x)\right)\in L^1(\mathbb{R}^d)$, then the unique stationary solution to the density given by the Fokker Planck equation* (6) *is*

$$\mathrm{p}^\infty(x) := \frac{1}{Z}\exp\left(-\frac{2}{\sigma^2}\bigg(w(x)\bigg)\right)(x), \tag{8}$$

where $Z=\int\exp\left(-\frac{2}{\sigma^2}w(x)\right)\,\mathrm{d}x$.

Proof. We observe that the (6) is the Smoluchowski equation for an overdamped Langevin system given by

$$\mathrm{d}x_s = -\nabla(\nu+v^\infty/R)(x_s)\,\mathrm{d}s+\sigma\mathrm{d}w_s. \tag{9}$$

Under the assumptions above, the proof then follows directly from proposition 4.2, pp 110 in [26]. □

Decay of an initial density of particles under uncontrolled (or open loop) overdamped Langevin dynamics to a stationary density is a classical topic [27]. We address the question of decay of a locally perturbed density of agents in a MFG to a steady state density under the closed loop time varying as well as steady state MFG optimal controls. The perturbation analysis then corresponds to a fully coupled forward-backward PDE system. The proposed approach leads to a general method to obtain stability constraints on the *control design parameters*, with explicit analytical results in certain cases.

To derive the linearization of MFG system (5, 6) around the pair $(v^\infty,\mathrm{p}^\infty)$, we write the perturbed density and value functions as $\mathrm{p}(t,x)=\mathrm{p}^\infty(x)(1+\epsilon\tilde{\mathrm{p}}(x,t))$, and $v(t,x)=v^\infty(x)+\epsilon\tilde{v}(x,t)$ respectively. The corresponding perturbed cost is $q(x,\mathrm{p})=q(x,\mathrm{p}^\infty(\cdot))+\epsilon\tilde{q}(x,\mathrm{p}^\infty(\cdot),\tilde{\mathrm{p}}(t,\cdot))$ where $\epsilon>0$. We use $q^\infty(x):=q(x,\mathrm{p}^\infty(\cdot))$, and $\tilde{q}(x):=\tilde{q}(x,\mathrm{p}^\infty(\cdot),\tilde{\mathrm{p}}(t,\cdot))$ for brevity.

The generator of a Langevin process is intrinsically linked to the stability properties of its density dynamics. We denote the generator of the optimally controlled agent dynamics (9) as $\mathcal{L}(\cdot):=-\nabla(\nu+v^\infty/R)\cdot\nabla(\cdot)+(\sigma^2/2)\Delta(\cdot)$ and its $L^2(\mathbb{R})$ adjoint $\mathcal{L}^\dagger(\cdot):=\nabla\cdot(\nabla(\nu+v^\infty/R)(\cdot))+(\sigma^2/2)\Delta(\cdot)$.

Theorem III.2. *If $(v^\infty(x),\mathrm{p}^\infty(x))$ are smooth steady state solutions to the MF system (5, 6) wherein ν is a smooth function such that $\lim_{|x|\to+\infty} w(x)=+\infty$ and $\exp\left(-\frac{2}{\sigma^2}w(x)\right)\in L^1(\mathbb{R}^d)$, then the linearization of the MF system (3, 4) around $(v^\infty(x),\mathrm{p}^\infty(x))$ for all $(t,x)\in[0,+\infty)\times\mathbb{R}^d$ is*

$$-\partial_t\tilde{v} = \tilde{q}-\rho\tilde{v}+\mathcal{L}\tilde{v}, \tag{10}$$

$$\partial_t\tilde{\mathrm{p}} = (2/\sigma^2R)\mathcal{L}\tilde{v}+\mathcal{L}\tilde{\mathrm{p}}, \tag{11}$$

where $\tilde{\mathrm{p}}(0,x)$ is given, $\int_{\mathbb{R}^d}\mathrm{p}^\infty(x)(1+\epsilon\tilde{\mathrm{p}}(t,x))\mathrm{d}x=1$ for all $t\geq 0,\epsilon>0$, $\lim_{|x|\to+\infty}\tilde{\mathrm{p}}(t,x)=0$ for all $t\geq 0$ and $\lim_{t\to+\infty}\mathrm{e}^{-\rho t}\tilde{v}(t,x_t)=0$.

Proof. Substituting the perturbation density $\mathrm{p}=\mathrm{p}^\infty(1+\epsilon\tilde{\mathrm{p}})$ in (4), using the fixed point equation (6) and neglecting higher order ϵ terms we have

$$\begin{aligned}\epsilon\partial_t(\mathrm{p}^\infty\tilde{\mathrm{p}}) =& \epsilon\nabla\cdot\left(\frac{\nabla\tilde{v}}{R}\mathrm{p}^\infty+\nabla(\nu+\frac{v^\infty}{R})\mathrm{p}^\infty\tilde{\mathrm{p}}\right)\\ &+\epsilon\frac{\sigma^2}{2}\Delta(\mathrm{p}^\infty\tilde{\mathrm{p}}),\end{aligned} \tag{12}$$

so that using the operator $\mathcal{L}^\dagger$ and the fact that $\epsilon>0$,

$$\partial_t(\mathrm{p}^\infty\tilde{\mathrm{p}}) = \nabla\cdot\bigg(\nabla\tilde{v}\;\mathrm{p}^\infty/R\bigg)+\mathcal{L}^\dagger(\mathrm{p}^\infty\tilde{\mathrm{p}}). \tag{13}$$

It can be verified [28] that for a smooth function $f(x)$ the generator and its adjoint operator satisfy the *detailed balance* property $\mathcal{L}^\dagger(\mathrm{p}^\infty f)=\mathrm{p}^\infty\mathcal{L}f$. Therefore from (13) we have

$$\mathrm{p}^\infty\partial_t\tilde{\mathrm{p}} = \frac{1}{R}\bigg(\mathrm{p}^\infty\Delta\tilde{v}+\nabla\tilde{v}\cdot\nabla\mathrm{p}^\infty\bigg)+\mathrm{p}^\infty\mathcal{L}\tilde{\mathrm{p}}. \tag{14}$$

From the assumed conditions on the potential and value functions, lemma III.1 gives the stationary density, so that $\nabla\mathrm{p}^\infty=-\frac{2}{\sigma^2}(\nabla\nu+\frac{\nabla v^\infty}{R})\mathrm{p}^\infty$. Then the previous equation simplifies as

$$\mathrm{p}^\infty\partial_t\tilde{\mathrm{p}} = \mathrm{p}^\infty\bigg(\mathcal{L}\tilde{\mathrm{p}}+\frac{2}{\sigma^2R}\mathcal{L}\tilde{v}\bigg), \tag{15}$$

giving us the density perturbation equation since $\mathrm{p}^\infty(x)>0$. Substituting the perturbation value function $v=v^\infty+\epsilon\tilde{v}$ in (3), using the fixed point equation (5) and neglecting higher order ϵ terms gives

$$\begin{aligned}-\epsilon\partial_t\tilde{v} =& \epsilon\tilde{q}-\epsilon\rho\tilde{v}-\epsilon\nabla\left(\frac{v^\infty}{R}+\nu\right)\cdot\nabla\tilde{v}\\ &+\epsilon\frac{\sigma^2}{2}\Delta\tilde{v}.\end{aligned} \tag{16}$$

Using the operator definition and since $\epsilon>0$ we get the required result. The mass conservation and boundary constraints on $\tilde{v},\tilde{p}$ follow directly from those constraints on (3, 4). □

In the following two sections we will apply the above result to obtain stability results for two MFG models. Note that the perturbation system may be written in concatenated form as

$$\partial_t\begin{bmatrix}\tilde{v}\\ \tilde{\mathrm{p}}\end{bmatrix} = \begin{bmatrix}-\mathcal{L}+\rho & 0\\ \frac{2}{\sigma^2R}\mathcal{L} & \mathcal{L}\end{bmatrix}\begin{bmatrix}\tilde{v}\\ \tilde{\mathrm{p}}\end{bmatrix}+\begin{bmatrix}-\tilde{q}\\ 0\end{bmatrix}. \tag{17}$$

IV. A Population Distribution model

We present the linear stability result for a population distribution MFG model in this section. A cost function with local density dependence is used in this model to mimic a population of agents with identical dynamics, seeking to minimize their cost functional but with a preference for imitating their peers. This models agents in an economic network [29]. A reference case for this model is [11] where the simplest case of integrator agent dynamics was treated. Note that while a strictly increasing cost function $q(p(t,\cdot))$ models aversion among agents, a strictly decreasing one models cohesion [30]. We consider a model comprised by the OCP (2) with the negative log density cost and agents following nonlinear Langevin dynamics (1). Since this cost function is strictly decreasing in the density, it models agents which are locally cohesive in the sense that they want to resemble their peers as much as possible [31].

The MF optimality system for this model consists of the coupled system (3, 4) along with the cost coupling equation

$$q(x,\mathrm{p}(t,\cdot)) = -\ln\mathrm{p}(t,x), \tag{18}$$

where $\mathrm{p}(0,x)=\mathrm{p}_0(x)$ is the given initial density of agents, $\int\mathrm{p}(t,x)\mathrm{d}x=1$ for all $t\geq 0$, $\lim_{t\to+\infty}\mathrm{p}(t,x)=0$ and $\lim_{t\to+\infty}\mathrm{e}^{-\rho t}v(t,x_t)=0$.

A. Stationary Solution

The stationary MF optimality system is given by (5, 6) and the cost coupling equation

$$q^{\infty}(x) = -\ln \mathrm{p}^{\infty}(x), \tag{19}$$

where $\int \mathrm{p}^{\infty}(x)\mathrm{d}x = 1$ and $\lim\limits_{t\to+\infty} \mathrm{e}^{-\rho t} v^{\infty}(x_t) = 0$.

Calculating analytical solutions to HJB PDEs is a daunting task, examples of which are rare and mainly related to linear-quadratic regimes. The presented approach aims at being applicable to the most general class of nonlinear dynamics. We show that under certain conditions on the (unknown) stationary solution $(v^{\infty}, \mathrm{p}^{\infty})$, one may obtain sufficiency conditions required for linear stability of the population model. Conditions on the stationary solution required to guarantee stability are stated in the following assumptions. Let $w(x) := \nu(x) + \frac{v^{\infty}(x)}{R}$.

(A1) There exist $(v^{\infty}(x), p^{\infty}(x)) \in (C^2(\mathbb{R}^d))^2$ satisfying (5,6,19) such that $\lim\limits_{|x|\to+\infty} w(x) = +\infty$ and $\exp\left(-\frac{2}{\sigma^2} w(x)\right) \in L^1(\mathbb{R}^d)$.

Due to this assumption, lemma III.1 implies that the stationary density is uniquely determined by the analytical expression (8). The above assumption of radial unboundedness on the net potential $w(x)$ of the closed loop agent dynamics is readily satisfied in the case of controlled gradient flows appearing in physical systems [15] and stochastic gradient descent [18] since the Langevin potential itself satisfies this condition.

B. Linear Stability

Under the assumption **(A1)**, the perturbation PDEs for the value and density functions as well as the constraints follow directly from theorem III.2. The only term in (17) specific to the cost coupling (18) is given by

$$\tilde{q}(x; \tilde{\mathrm{p}}(t,\cdot)) = -\,\tilde{\mathrm{p}}(t,x), \tag{20}$$

using the Taylor series expansion.

We define a Hilbert space and a class of perturbations in it, for which we show stability.

Definition IV.1. *Let **(A1)** hold. Denote the density* $\mathrm{p}^{\infty}(x) := \frac{1}{Z}\exp\left(-\frac{2}{\sigma^2} w(x)\right)(x)$ *with the normalizing constant Z where (v^{∞}, p^{∞}) is a pair satisfying **(A1)**. Denote by $\mathcal{H}$ the Hilbert space $L^2(\mathrm{p}^{\infty}(x)\mathrm{d}x; \mathbb{R}^d)$. The class of mass preserving density perturbations is defined as* $S_0 := \left\{ q(x) \in \mathcal{H} \middle| \langle 1, q(x)\rangle_{\mathcal{H}} = 0 \right\}$.

Definition IV.2. *Let us denote the set of initial perturbed densities by* $S(\epsilon) = \left\{ p(0,x) = p^{\infty}(x)(1+\epsilon\tilde{\mathrm{p}}(0,x)) \middle| p(0,x) \geq 0, \tilde{\mathrm{p}}(0,x) \in S_0 \right\}$. *We say the fixed point $(v^{\infty}(x), p^{\infty}(x))$ of the MF optimality system (3, 4) is linearly asymptotically stable with respect to $S(\epsilon)$ if there exists a solution $(\tilde{v}(t,x), \tilde{p}(t,x))$ to the perturbation system (10,11) such that* $\lim\limits_{t\to+\infty} ||\tilde{\mathrm{p}}(t,x)||_{\mathcal{H}} = 0$.

Since we are concerned with stability of isolated fixed points, we *do not* assume that initial perturbations are mean preserving, in contrast to previous work on this topic [11].

Lemma IV.1. *If **(A1)** is true then $\mathcal{L}$ is self adjoint in $L^2(\mathrm{p}^{\infty}\mathrm{d}x; \mathbb{R}^d)$, negative semidefinite and its kernel consists of constants.*

Proof. Due to **(A1)** $v^{\infty}(x)$ is differentiable and the operator $\mathcal{L}$ is well defined. We observe that it is the generator of an overdamped Langevin system (9) under a potential $\nu + v^{\infty}/R$ and noise intensity σ. The proof follows from proposition 4.3, pp 111 in [26]. □

We need an assumption to obtain relevant properties of the generator of the controlled process.

(A2) $\lim\limits_{|x|\to+\infty} \left(\frac{|\nabla w(x)|^2}{2} - \frac{\sigma^2}{2}\Delta w(x) \right) = +\infty$ and $\nu(x) \in C^2(\mathbb{R}^d)$.

An example of an one dimensional MFG model with integrator dynamics and its corresponding stationary solution $v^{\infty}(x)$ satisfying this assumption was explicitly constructed in [11]. Using the stationary HJB equation (5) satisfied by v^{∞}, the above condition reduces to a condition on the cost function $q(x, \mathrm{p}^{\infty}) + (R/2)(\nabla\nu)^2 - (\sigma^2 R/2)\Delta\nu \xrightarrow[|x|\to+\infty]{} +\infty$.

Lemma IV.2. *Let **(A1, A2)** hold. Then $\mathrm{p}^{\infty}(x)$ satisfying **(A1)** and given by* (8)*, satisfies the Poincaré inequality with $\lambda > 0$, that is, there exists $\lambda > 0$ such that for all functions $f \in C^1(\mathbb{R}^d) \cap L^2(\mathrm{p}^{\infty}(x)\mathrm{d}x; \mathbb{R}^d)$ and $\int f \mathrm{p}^{\infty}(x)\mathrm{d}x = 0$, we have*

$$\begin{aligned} &\lambda \frac{2}{\sigma^2} ||f||^2_{L^2(\mathrm{p}^{\infty}(x)\mathrm{d}x;\mathbb{R}^d)} \\ &\leq ||\nabla f||_{L^2(\mathrm{p}^{\infty}(x);\mathbb{R}^d)} = -\langle \mathcal{L}f, f\rangle_{L^2(\mathrm{p}^{\infty}(x);\mathbb{R}^d)}. \end{aligned} \tag{21}$$

Proof. The assumptions imply that $v^{\infty}(x) \in C^2(\mathbb{R}^d)$, and hence, $(\nu + v^{\infty}/R)(\cdot) \in C^2(\mathbb{R}^d)$. Observe that operator $\mathcal{L}$ is the generator of an overdamped Langevin system under a potential $\nu + v^{\infty}/R$ and noise intensity σ. The proof then follows from theorem 4.3, pp 112 in [26]. □

Lemma IV.1 implies that eigenvalues of $\mathcal{L}$ are real, negative semidefinite and its eigenfunctions are orthonormal in $L^2(\mathrm{p}^{\infty}(x)\mathrm{d}x; \mathbb{R}^d)$ while lemma IV.2 implies that the eigenvalues of $\mathcal{L}$ are discrete and its eigenfunctions are complete on $L^2(\mathrm{p}^{\infty}(x)\mathrm{d}x; \mathbb{R}^d)$ [28]. We denote the eigenvalues $\{\xi_n\}_{n\geq 0}$ and corresponding eigenfunctions $\{\Xi_n\}_{n\geq 0}$ of $\mathcal{L}$ which form a complete orthonormal basis of $\mathcal{H}$. Let eigenvalues $\{\xi_n\}_{n\geq 0}$ be indexed in descending order of magnitude $0 = \xi_0 > \xi_1 > \ldots > \xi_n > \ldots$ and let $\Xi_0 = 1$.

Remark 2. *The detailed balance $\mathcal{L}^{\dagger}(\mathrm{p}^{\infty} f) = \mathrm{p}^{\infty}\mathcal{L}(f)$ used in proof of theorem* (III.2) *is the key property, because of which we have distinct, real and non negative eigenvalues [28] of the generator $\mathcal{L}$. These eigen properties make the presented approach to stability analysis of MFGs possible, through the result in theorem III.2.*

(A3) $\rho - \frac{2}{\sigma^2 R} > \xi_n$ for all $n \geq 1$.

The assumption above is the explicit control design constraint required to show stability. Denote the matrix associated with the MF system for the population distribution model $A_n := \begin{bmatrix} -\xi_n+\rho & 1 \\ \frac{2}{\sigma^2 R}\xi_n & \xi_n \end{bmatrix}$.

Lemma IV.3. *If* $\xi_n \neq 0$ *and* $\rho - \frac{2}{\sigma^2 R} > \xi_n$ *then the eigenvalues of* A_n *are real, distinct and ordered* $\lambda_n^1 < 0 < \lambda_n^2$.

Proof. The characteristic equation of A_n is $(\lambda_n)^2 - \rho\lambda_n + (\rho - \xi_n)\lambda_n + \frac{2}{\sigma^2 R}\xi_n = 0$ has the eigenvalue roots $\lambda_n^{1,2} = \frac{\rho}{2} \pm \sqrt{\left(\frac{\rho}{2}\right)^2 - (\rho-\xi_n)\xi_n + \frac{2}{\sigma^2 R}\xi_n}$ from which the result follows. □

The spectral properties of perturbation MFG system derived in this section allow us to extend the methods in [11] (applied to integrator agent dynamics) to the case of nonlinear Langevin agent dynamics. Note that the stationary solution as well as the eigenbasis are not explicitly known here, unlike in previous works which exploit the Hermite basis resulting from explicitly known quadratic-Gaussian stationary solutions.

Theorem IV.4. *Let (**A1**, **A2**, **A3**) hold, and* $(v^\infty(x), \mathrm{p}^\infty(x))$ *be a stationary solution to the MF system (3, 4, 18). If perturbation* $\tilde{\mathrm{p}}(0,x) \in S_0$ *and* $\{v_n, p_n\}_{n\geq 0}$ *is determined by* $p_0(t) = 0$*, and for* $n \geq 0$

$$\begin{bmatrix} \dot{v_n} \\ \dot{p_n} \end{bmatrix} = A_n \begin{bmatrix} v_n \\ p_n \end{bmatrix}, \tag{22}$$

$$\mathrm{p}_n(0) = \langle \tilde{\mathrm{p}}(0,x), \Xi_n(x) \rangle_{\mathcal{H}}, \tag{23}$$

then $\{\tilde{v}(t,x) = \sum_{n=0}^{+\infty} v_n(t)\Xi_n(x),\ \tilde{\mathrm{p}}(t,x) = \sum_{n=0}^{+\infty} p_n(t)\Xi_n(x)\}$ *are unique* $\mathcal{H}$ *solutions to the perturbation MF system (10,11,20).* $\mathrm{p}^\infty(x)$ *is linearly asymptotically stable with respect to* $S(\epsilon)$.

Proof. ***Finite time solution:*** We first construct finite time solutions to the perturbation system (10,11,20) under initial and terminal time boundary conditions $\tilde{v}(T,x) \in \mathcal{H}$, $\tilde{\mathrm{p}}(0,x) \in S_0$. We have the unique representations $\tilde{v}(T,x) = \sum_{n=0}^{+\infty} v_n(T)\Xi_n(x)$ and $\tilde{\mathrm{p}}(0,x) = \sum_{n=0}^{+\infty} p_n(0)\Xi_n(x)$, where

$$v_n(T)_{n\geq 0} = \langle \tilde{v}(T,x), \Xi_n(x) \rangle_{\mathcal{H}}, \tag{24}$$

and $p_n(0)_{n\geq 0}$ is given by (23).

Consider the infinite sums $\{\sum_{n=0}^{+\infty} v_n(t)\Xi_n(x), \sum_{n=0}^{+\infty} p_n(t)\Xi_n(x)\}$. Using the eigen property $\mathcal{L}\Xi_n(x) = \xi_n\Xi_n(x)$, and inserting the infinite sums into the perturbation system (10, 11, 20) yields the ODE system (22).

For $n = 0$, since $\tilde{\mathrm{p}}(0,x) \in S_0$ and $\Xi_0 = 1$, we know that $p_0(0) = \langle \Xi_0, \tilde{p}(0,x) \rangle = 0$. Since $\xi_0 = 0$, from the matrix A_n we have $\dot{p}_0(t) = 0$ implying $p_0(t) = 0$ for all $t \in [0,T]$. Therefore, $v_0(t) = v_0(T)\mathrm{e}^{-\rho(T-t)}$.

For $n \geq 1$, from lemma IV.3 we have that the eigenvalues $spec(A_n) = \lambda_n^{1,2}$ are distinct, real and are ordered $\lambda_n^1 < 0 < \lambda_n^2$. We may write

$$\begin{bmatrix} v_n(t) \\ p_n(t) \end{bmatrix} = C_1^{n,T}\mathrm{e}^{\lambda_n^1 t}\begin{bmatrix} 1 \\ e_n^1 \end{bmatrix} + C_2^{n,T}\mathrm{e}^{\lambda_n^2 t}\begin{bmatrix} 1 \\ e_n^2 \end{bmatrix}, \tag{25}$$

with eigenvector components $e_n^{1,2} = \xi_n - \rho + \lambda_n^{1,2}$. Boundary conditions give us $v_n(T) = C_1^{n,T}\mathrm{e}^{\lambda_n^1 T} + C_2^{n,T}\mathrm{e}^{\lambda_n^2 T}$ and $p_n(0) = C_1^{n,T}e_n^1 + C_2^{n,T}e_n^2$ implying

$$C_1^{n,T} = \frac{(e_n^2/e_n^1)v_n(T) - \mathrm{e}^{\lambda_n^2 T}(p_n(0)/e_n^1)}{(e_n^2/e_n^1)\mathrm{e}^{\lambda_n^1 T} - \mathrm{e}^{\lambda_n^2 T}}, \tag{26}$$

$$C_2^{n,T} = \frac{(p_n(0)/e_n^1) - v_n(T)}{(e_n^2/e_n^1)\mathrm{e}^{\lambda_n^1 T} - \mathrm{e}^{\lambda_n^2 T}}. \tag{27}$$

From the eigenvalues given by lemma IV.3 and since in the limit $\xi_n \to -\infty$, we observe that $e_n^1 \sim -2|\xi_n|$ and $e_n^2 \sim \frac{\rho}{2}$ as $n \to +\infty$ so that in the limit $C_1^{n,T} \sim \frac{p_n(0)}{e_n^1}$ and $C_2^{n,T} \sim \frac{v_n(T)}{\mathrm{e}^{\lambda_n^2 T}}$. We can therefore say that $v_n(t) = O\left(-\frac{p_n(0)}{2|\xi_n|}\mathrm{e}^{-\lambda_n^1 t}\right) + O\left(v_n(T)\mathrm{e}^{-\lambda_n^2(T-t)}\right)$ and $p_n(t) = O\left(p_n(0)\mathrm{e}^{\lambda_n^1 t}\right) + O\left(v_n(T)\mathrm{e}^{-\lambda_n^2(T-t)}\right)$. From these estimates we can say that $\sum_{n=0}^{+\infty} v_n(t)\Xi_n(x)$ and $\sum_{n=0}^{+\infty} p_n(t)\Xi_n(x)$ given by the ODE system (22, 24, 23) are in $C^\infty([0,T]\times\mathbb{R}^d)$ and $\mathcal{H}$.

Since $\{\Xi_n\}_{n\geq 0}$ is a complete basis to $\mathcal{H}$, any solution in $\mathcal{H}$ to the system (10, 11, 20) must have the form $\{\tilde{v}(t,x) = \sum_{n=0}^{+\infty} v_n(t)\Xi_n(x), \tilde{p}(t,x) = \sum_{n=0}^{+\infty} p_n(t)\Xi_n(x)\}$ where $\{v_n, p_n\}_{n\geq 0}$ are finite for all $t \in [0,T]$. This concludes the proof that such a $\{\tilde{v}(t,x), \tilde{\mathrm{p}}(t,x)\}$ governed by the ODE system (22, 23, 24) is a unique $\mathcal{H}$ solution to the perturbation system (10, 11, 20).

Asymptotic stability: Now, we construct infinite time solutions by considering the limit $T \to +\infty$ of the solutions in the finite time case. As explained in the finite time solutions case, it can be shown that $p_0(t) = 0$ at all times.

The pair $\{\tilde{v}(t,x) = \sum_{n=0}^{+\infty} v_n(t)\Xi_n(x), \tilde{\mathrm{p}}(t,x) = \sum_{n=0}^{+\infty} p_n(t)\Xi_n(x)\}$ is a unique solution specified by (25) given the initial and terminal coefficients $\mathrm{p}_n(0)$ and $v_n(T)$ for all $n \geq 0$. Now, if $\tilde{v}(t,x) \in \mathcal{H}$ then $\lim_{t\to+\infty}|v_n(t)| < +\infty$ for all $n \geq 0$. It is also known that $|p_n(0)| < +\infty$. Therefore, for $n = 0$, this means that $p_0(t) = 0$ and $v_0(t) = v_0(T)\mathrm{e}^{-\rho(T-t)} \xrightarrow{T\to+\infty} 0$.

From lemma IV.3, the eigenvalues of A_n are ordered $\lambda_n^1 < 0 < \lambda_n^2$ for all $n \geq 1$ due to **(A3)**. Therefore, for all $n \geq 1$, we observe from the finite time solutions (26, 27) to the ODE system (22), that $C_1^{n,T} \to \frac{p_n(0)}{e_n^1}$ and $C_2^{n,T} \to v_n(T)\mathrm{e}^{-\lambda_n^2 T}$ as $T \to +\infty$. Since $\lambda_n^1 < 0 < \lambda_n^2$, for any $\alpha \in (0, \frac{1}{2})$ and as $T \to +\infty$, it can be obtained from (25) that

$$\begin{aligned} \sup_{t\in[\alpha T,(1-\alpha T)]} |v_n(t)| &\leq |C_1^{n,T}|\mathrm{e}^{\lambda_n^1 \alpha T} + |C_2^{n,T}|\mathrm{e}^{\lambda_n^2(1-\alpha)T} \\ &\leq \left|\frac{p_n(0)}{e_n^1}\right|\mathrm{e}^{\lambda_n^1\alpha T} + |v_n(T)|\mathrm{e}^{-\lambda_n^2\alpha T}, \end{aligned} \tag{28}$$

$$\begin{aligned} \sup_{t\in[\alpha T,(1-\alpha T)]} |p_n(t)| &\leq |C_1^{n,T}||e_1^1|\mathrm{e}^{\lambda_n^1\alpha T} + |C_2^{n,T}||e_1^2|\mathrm{e}^{\lambda_n^2(1-\alpha)T} \\ &\leq |p_n(0)|\mathrm{e}^{\lambda_1^1\alpha T} + |v_n(T)|\mathrm{e}^{-\lambda_1^2\alpha T}, \end{aligned} \tag{29}$$

the right sides of which vanish in the limit since $|v_n(T)| < +\infty$ and $|p_n(0)| < +\infty$.

We have shown that the unique solution in $\mathcal{H}$ to the MF perturbation system has the properties $v_0(t) = p_0(t) = 0$, $\lim_{t\to+\infty} v_n(t) = 0$ and $\lim_{t\to+\infty} p_n(t) = 0$ for all $n \geq 1$.

Therefore using Parseval's theorem $||\tilde{v}(t,x)||_{L^2(\mathrm{p}^\infty(x);\mathbb{R}^d)} = \left(\sum_{n=1}^{+\infty} v_n(t)\right)^{\frac{1}{2}}$, $||\tilde{\mathrm{p}}(t,x)||_{L^2(\mathrm{p}^\infty(x);\mathbb{R}^d)} = \left(\sum_{n=1}^{+\infty} p_n^2(t)\right)^{\frac{1}{2}}$ and the Lebesgue dominated convergence theorem, we have that $\mathrm{p}^\infty(x)$ is linearly asymptotically stable with respect to perturbing densities in $S(\epsilon)$. □

V. A Mean Consensus Model

In this section we obtain stability results for a mean consensus MFG model using theorem (III.2). The model consists of the problem statement (2) with the *nonlocal* consensus cost $q(x,\mathrm{p}(t,\cdot)) = \frac{1}{2}\left(\int (x-x')\mathrm{p}(t,x')\,\mathrm{d}x'\right)^2$ and agents following controlled one dimensional Langevin dynamics (1) with quadratic restoring potential $\nu = \frac{1}{2}ax^2$, $a \neq 0$. A MFG model with consensus cost has been previously studied in [13] wherein it is assumed that all agents follow integrator dynamics, that is, the case $a = 0$. Although a more general potential $\nu(x)$ can be treated using the result (III.2) to obtain stability results, we choose to present the generalization only to the quadratic potential. This choice allows us to obtain analytical fixed point solutions for the stationary MF system, inspired by related work in [31] where fixed points solutions were found for a different class of MFGs. The linearity in passive agent dynamics also allows for mean consensus, as discussed later in this section.

The MF optimality system for this model consists of the coupled system (3, 4) wherein $\nu = \frac{1}{2}ax^2$, along with the cost coupling equation

$$q(x,\mathrm{p}(t,\cdot)) = \frac{1}{2}\left(\int (x-x')\mathrm{p}(t,x')\,\mathrm{d}x'\right)^2 \tag{30}$$

where $\mathrm{p}(0,x) = \mathrm{p}_0(x)$ is the given initial density of agents, $\int \mathrm{p}(t,x)\mathrm{d}x = 1$ for all $t \geq 0$, $\lim_{|x|\to+\infty} \mathrm{p}(t,x) = 0$ and $\lim_{t\to+\infty} \mathrm{e}^{-\rho t} v(t,x_t) = 0$.

A. Gaussian Stationary Solution

The stationary MF optimality system for this model consists of (5, 6) wherein $\nu = \frac{1}{2}ax^2$, along with the cost coupling equation

$$q^\infty(x) = \frac{1}{2}\left(\int (x-x')\mathrm{p}^\infty(x')\mathrm{d}x'\right)^2 \tag{31}$$

where $\int \mathrm{p}^\infty(x)\mathrm{d}x = 1$ and $\lim_{t\to+\infty} \mathrm{e}^{-\rho t} v^\infty(x_t) = 0$. We denote $\mu^* := \int_{\mathbb{R}} x'\mathrm{p}^\infty(x')\mathrm{d}x'$. In this subsection we will obtain solutions of the form

$$v^\infty(x) = \frac{\eta}{2}x^2 + \beta x + \omega, \tag{32}$$

$$\mathrm{p}^\infty(x) = \frac{1}{\sqrt{2\pi s^2}}\mathrm{e}^{-\frac{(x-\mu^*)^2}{2s^2}}, \tag{33}$$

to the value and density functions in the coupled optimality system (5, 6, 31). Parameters η, β and ω can be obtained by substituting (32) into (5), using (31) and equating coefficients of powers of x:

$$\omega = \frac{1}{\rho}\left(\frac{1}{2}(\mu^*)^2 - \frac{\beta^2}{2R} + \frac{\sigma^2}{2}\eta\right), \tag{34}$$

$$\beta = \frac{-\mu^*}{\rho + \frac{\eta}{R} + a}, \tag{35}$$

$$\eta^2 + 2R(\rho/2 + a)\eta - R = 0. \tag{36}$$

These parameters must satisfy additional conditions related to the validity of the solution ansatz, namely, $s^2 > 0$ and $v^\infty(x) > 0$ for all $x \in \mathbb{R}$. The unique positive solution to the Algebraic Riccati Equation (ARE) (36) which permits $v^\infty(x) > 0$ for all $x \in \mathbb{R}$ is

$$\eta = -R\left(\frac{\rho}{2} + a\right) + \sqrt{R^2\left(\frac{\rho}{2} + a\right)^2 + R}. \tag{37}$$

Choosing this solution, it is easily verified that $\rho + \frac{\eta}{R} + a > 0$. Equating our stationary density ansatz (33) with the unique Gibbs distribution solution (8) from lemma III.1 implies

$$\mu^* = \frac{-\beta}{(aR + \eta)}, \tag{38}$$

$$s^2 = \frac{\sigma^2}{2(a + \frac{\eta}{R})}. \tag{39}$$

Equations (35) and (38) are compatible only if $\mu^* = 0$ or $\frac{1}{\rho + \frac{\eta}{R} + a} = aR + \eta$. Using the ARE (36) it can be verified that the latter condition is equivalent to $a = -\rho$. We conclude that the Gaussian stationary solutions can be categorized into two distinct cases depending upon problem parameters: (1) if $a \neq -\rho$, there exists a unique solution with $\mu^* = 0$ and (2) if $a = -\rho$, there exist a continuum of solutions, since $\mu^* \in \mathbb{R}$ can be chosen arbitrarily. The following assumption is needed to ensure $s^2 > 0$.

(B1) $a + \frac{\eta}{R} > 0$ for all $a \neq 0$.

Given a value of a, we provide the range of control design parameters for which **(B1)** is true in the following lemma, which can be verified by substitution in equation (37).

Lemma V.1. *Let $a_{l,u} := \frac{-\rho}{2} \pm \sqrt{\left(\frac{\rho}{2}\right)^2 - \frac{1}{R}}$. Then **(B1)** holds if either*

- $\rho < \frac{2}{\sqrt{R}}$
  or
- $\rho > \frac{2}{\sqrt{R}}$ *and* $a \in (-\infty, a_u) \cup (a_l, +\infty)$.

We summarize the obtained quadratic-Gaussian solution to the stationary MF system below.

Lemma V.2. *Let **(B1)** hold. 1) Case $a \neq -\rho$: The unique quadratic-Gaussian solution to the stationary MF optimality system (5, 6) (with $\nu(x) = \frac{1}{2}ax^2, a \neq 0$) is the pair $(v^\infty(x) = \frac{\eta}{2}x^2 + \frac{\sigma^2\eta}{2\rho}, \mathrm{p}^\infty(x) = \frac{1}{\sqrt{2\pi s^2}}\mathrm{e}^{-\frac{x^2}{2s^2}})$ where (η, s) are defined by (37, 39). Furthermore, $q^\infty(x) = \frac{1}{2}x^2$.*
2) Case $a = -\rho$: For each $\mu^* \in \mathbb{R}$, there exists a pair $(v^\infty(x), \mathrm{p}^\infty(x))$ given by equations (32, 33) that is a solution to the stationary MF optimality system (5, 6) (with $\nu(x) = \frac{1}{2}ax^2, a \neq 0$). The parameters (ω, β, η, s) are given by equations (34, 35, 37, 39). Furthermore, $q^\infty(x) = \frac{1}{2}(x-\mu^*)^2$.

Proof. In both cases, $q^{\infty}(x) = \frac{1}{2}(x-\mu^*)^2$ follows from equation (31) and assumption **(B1)** ensures that $s^2 > 0$ in the unique Gaussian Gibbs distribution (33) corresponding to the quadratic value function (32).

In case 1, the solution to the stationary value function is obtained by substituting $\mu^* = 0$ in equations (34), (35). This completes the first part of the proof. In case 2, for a given value of $\mu^* \in \mathbb{R}$, the solution to the value function maybe obtained similarly to the previous case.

From the expression for the Gibbs distribution (8) and equation (33) we have $\frac{2}{\sigma^2}v^{\infty}(x) = \frac{(x-\mu^*)^2}{s^2} \geq 0$, which concludes the proof. $\square$

B. Linear Stability

We define a Hilbert space and a class perturbations in it, for which we show stability.

Definition V.1. *Denote Gaussian density* $\mathrm{p}_G^{\infty}(x) := \frac{1}{\sqrt{2\pi s^2}}\mathrm{e}^{-\frac{(x-\mu^*)^2}{2s^2}}$ *with* $\mu^* \in \mathbb{R}$, $s^2 > 0$. *Denote by* $\mathcal{H}_G$ *the Hilbert space* $L^2(\mathrm{p}_G^{\infty}(x)\mathrm{d}x;\mathbb{R})$. *The class of mass preserving density perturbations is defined as* $S_0 := \left\{q(x) \in \mathcal{H} \middle| \langle 1, q(x)\rangle_{\mathcal{H}_G} = 0\right\}$. *The class of mass and mean preserving density perturbations is defined as* $S_1 := \left\{q(x) \middle| \langle 1, q(x)\rangle_{\mathcal{H}_G} = 0, \langle x, q(x)\rangle_{\mathcal{H}_G} = 0\right\}$.

The class of initial perturbed densities and linear asymptotic stability can be defined analogously from the previous section by replacing $\mathrm{p}^{\infty}(x)$ by $\mathrm{p}_G^{\infty}(x)$ in definition IV.2.

The lemma below follows from theorem III.2 and Taylor expansion of q in (30) around the fixed point.

Lemma V.3. *Let* $\nu(x) = \frac{1}{2}ax^2$, $a \neq 0$. *If* ***(B1)*** *holds, and* ($v^{\infty}(x)$, $\mathrm{p}^{\infty}(x)$, $q^{\infty}(x)$) *given by lemma V.2 is a stationary solution to the nonlinear MF system (5, 6, 31) then the linearization of the system around this solution for all* $(t,x) \in [0,+\infty) \times \mathbb{R}$ *is given by (10,11) and*

$$\tilde{q}(x) = -(x-\mu^*)\left(\int_{\mathbb{R}} x'\mathrm{p}^{\infty}(x')\tilde{\mathrm{p}}(t,x')\mathrm{d}x'\right), \tag{40}$$

where $\tilde{\mathrm{p}}(0,x)$ *is given,* $\int_{\mathbb{R}} \mathrm{p}^{\infty}(x)(1+\epsilon\tilde{\mathrm{p}}(t,x))\mathrm{d}x = 1$ *for all* $t \geq 0, \epsilon > 0$, $\lim_{|x|\to+\infty} \tilde{\mathrm{p}}(t,x) = 0$ *for all* $t \geq 0$ *and* $\lim_{t\to+\infty} \mathrm{e}^{-\rho t}\tilde{v}(t,x_t) = 0$.

We now state eigen properties of the generator ([26], [32]) of the controlled process for the consensus model. We define normalized Hermite polynomials $\{H_n(x)\}_{n\in\mathbb{W}}$ for the space $L^2(\mathrm{p}_G^{\infty}\mathrm{d}x;\mathbb{R})$ as $H_n(x) = s^n\frac{1}{\sqrt{n!}}(-1)^n\mathrm{e}^{\frac{(x-\mu^*)^2}{2s^2}}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{-\frac{(x-\mu^*)^2}{2s^2}}$. These polynomials with $n \geq 0$, form a countable orthonormal basis of the space $\mathcal{H}_G$. $\{H_n(x)\}_{n\in\mathbb{W}}$ are eigenfunctions of the operator $\mathcal{L}$ wherein $\nu(x) = \frac{1}{2}ax^2$, with the [13] eigenproperty $\mathcal{L}H_n = -\frac{\sigma^2}{2s^2}nH_n = -(a+\frac{\eta}{R})nH_n$. The following condition is needed for stability of the consensus model.

(B2) $a(a+\rho) \geq 0$.

Note that this assumption is true if and only if $a \in (-\infty,-\rho] \cup (0,+\infty)$, recalling that $a \neq 0$. Denote the matrix associated with the MF system for the consensus model $B_n := \begin{bmatrix} \frac{\sigma^2 n}{2s^2}+\rho & s^2\delta(n-1) \\ \frac{-n}{s^2R} & \frac{-\sigma^2 n}{2s^2}\end{bmatrix}$.

Lemma V.4. *Let* ***(B1, B2)*** *hold. Then, for all* $n \geq 2$ *the eigenvalues of* B_n, $\lambda_n^{1,2} = \frac{\rho}{2} \pm \sqrt{\left(\frac{\rho}{2}\right)^2 + \frac{\sigma^2 n}{2s^2}\left(\frac{\sigma^2 n}{2s^2}+\rho\right)} = \{-\frac{\sigma^2 n}{2s^2}, \frac{\sigma^2 n}{2s^2}+\rho\} = \{-\left(a+\frac{\eta}{R}\right)n, \left(a+\frac{\eta}{R}\right)n+\rho\}$ *are real, distinct and ordered* $\lambda_n^1 < 0 < \lambda_n^2$. *Furthermore, the eigenvalues of* B_1 *denoted* $\lambda_1^{1,2}$ *are real, distinct and ordered* $\lambda_1^1 < 0 < \lambda_1^2$ *if* $a \in (-\infty,-\rho) \cup (0,+\infty)$ *and* $\lambda_1^{1,2} = \{0,\rho\}$ *if* $a = -\rho$.

On applying the ARE (36), we see that the eigenvalues $\lambda_1^{1,2} = \frac{\rho}{2} \pm \sqrt{\left(\frac{\rho}{2}\right)^2 + \frac{\sigma^2}{2s^2}\left(\frac{\sigma^2}{2s^2}+\rho-\frac{2s^2}{\sigma^2 R}\right)} = \{-a, a+\rho\}$. Choosing to denote the lower of the eigenvalues by λ_1^1, we see that $\lambda_1^1 = -a$ if $a \in (0,+\infty)$ and $\lambda_1^1 = a+\rho$ if $a \in (-\infty,-\rho)$.

Spectral properties of the perturbation MFG system obtained in this section allow us to generalize the methods in [13] (applied to integrator agent dynamics) to prove stability of fixed points for MFG with linear Langevin agent passive dynamics. In the following theorem, we show linear stability of unique zero mean stationary density ($\mu^* = 0$, corresponding to $a \neq -\rho$) with respect to mass preserving density perturbations ($\tilde{\mathrm{p}}(0,x) \in S_0$).

Theorem V.5. *Let* $\nu(x) = \frac{1}{2}ax^2, a \notin \{0,-\rho\}$. *Let* ***(B1, B2)*** *hold. Let* $(v^{\infty}(x), \mathrm{p}^{\infty}(x), q^{\infty}(x))$ *given by lemma V.2 be a stationary solution to the MF system (5, 6, 31). If perturbation* $\tilde{\mathrm{p}}(0,x) \in S_0$, *and* $\{v_n, p_n\}_{n\geq 0}$ *are determined by*

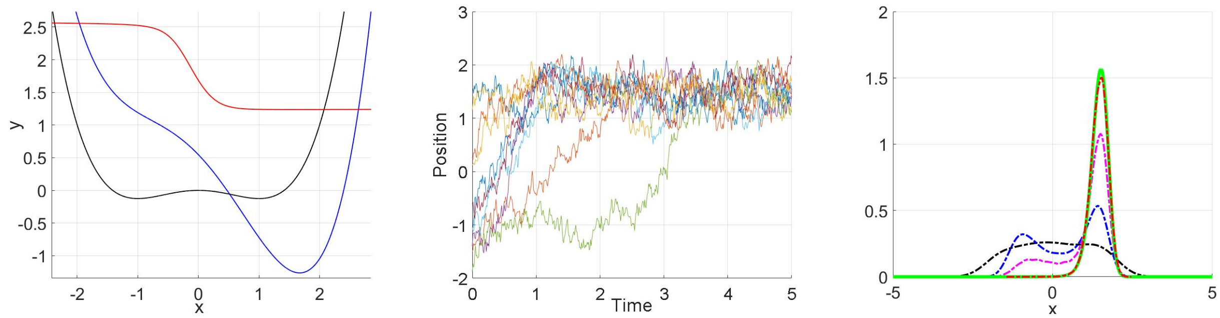
$$\begin{bmatrix}\dot{v}_n \\ \dot{p}_n\end{bmatrix} = B_n\begin{bmatrix}v_n \\ p_n\end{bmatrix}, \quad n \geq 0, \tag{41}$$

then $\tilde{v}(t,x) = \sum_{n=0}^{+\infty} v_n(t)H_n(x)$, $\tilde{\mathrm{p}}(t,x) = \sum_{n=0}^{+\infty} p_n(t)H_n(x)$ *are unique* $\mathcal{H}_G$ *solutions to the perturbation MF system (10, 11, 40). Moreover, the steady state density* $\mathrm{p}^{\infty}(x) = \mathrm{p}_G^{\infty}(x)$ *is linearly asyptotically stable with respect to* $S(\epsilon)$. *Furthermore,* $\tilde{\mathrm{p}}(t,x) = p_1(0)\mathrm{e}^{\lambda_1^1 t}H_1(x) + \sum_{n=2}^{+\infty} p_n(0)\mathrm{e}^{-\frac{-\sigma^2 n}{2s^2}t}H_n(x)$, $\tilde{q}(x;\tilde{\mathrm{p}}(t,x)) = -s^2 p_1(0)H_1(x) = 0$, *and* $\tilde{v}(t,x) = \frac{s^2 p_1(0)}{\frac{\sigma^2}{2s^2}+\rho-\lambda_1^1}\mathrm{e}^{\lambda_1^1 t}H_1(x)$ *where* λ_1^1 *is defined in lemma V.4.*

Proof. We construct $\mathcal{H}_G$ solutions of form $\tilde{v}(t,x) = \sum_{n=0}^{+\infty} v_n(t)H_n(x)$, $\tilde{\mathrm{p}}(t,x) = \sum_{n=0}^{+\infty} p_n(t)H_n(x)$ to the perturbation MF system (10, 11, 40) and show that they are unique. Since $\tilde{\mathrm{p}}(0,x) \in \mathcal{H}_G$ we have the unique representation $\tilde{\mathrm{p}}(0,x) = \sum_{n=0}^{+\infty} p_n(0)H_n(x)$.

Since $H_1(x) = \frac{x-\mu^*}{s}$, from (40) we note that $\tilde{q}(x;\tilde{\mathrm{p}}(t,x)) = -sH_1(x)\sum_{n=0}^{+\infty} p_n(t)\langle\tilde{\mathrm{p}}(t,x), x\rangle = -s^2 p_1(t)H_1(x)$. Substituting the selected form of the solutions into the perturbation system (10, 11, 40) and using the eigen property of the operator yields the ODEs (41).

(i) Case $n = 0$: Since $H_0(x) = 1$, and $\tilde{\mathrm{p}}(0,x) \in S_0$, we have $p_0(0) = \langle\tilde{\mathrm{p}}(0,x), 1\rangle_{\mathcal{H}_G} = 0$. Therefore, from the ODE system (41) and matrix B_n, we have $\dot{p}_0 = 0$ and $\dot{v}_0 = \rho v_0$ implying $p_0(t) = 0$ and $v_0(t) = v_0(0)\mathrm{e}^{\rho t}$ for all $t > 0$. So,



(a) (b) (c)

Fig. 1: (a) Bistable potential (black), $v^{\infty}(x)$ for population model (blue) and the consensus cost case (red). Population model, $\alpha = 0.5$, $\sigma = 1$, $\rho = 5$, $Q = 10$ and $R = 0.5$: (b) Stochastic paths for ten agents (c) Evolution of density at various times, $t = 0$ (black), $t = T/5$ (blue), $t = 2T/5$ (pink), $t = T$ (red) to the PDE solution (green)

the only solution allowing $\tilde{v}(t,x) \in \mathcal{H}_G$ is $v_0(t) = 0$.

(ii) Case $n = 1$: In this case, from (39),

$$B_1 = \begin{bmatrix} \left(a + \frac{\eta}{R} + \rho\right) & s^2 \\ -\frac{1}{s^2 R} & -\left(a + \frac{\eta}{R}\right) \end{bmatrix}. \tag{42}$$

The assumptions imply $a \in (-\infty, -\rho) \cup (0, +\infty)$. Hence, from lemma V.4, the eigenvalues $spec(B_1) = \lambda_1^{1,2}$ are ordered $\lambda_1^1 < 0 < \lambda_1^2$. Consider the finite time boundary conditions $p_1(0), v_1(T)$ to ODE system in this case. We may write

$$\begin{bmatrix} v_1(t) \\ p_1(t) \end{bmatrix} = C_1^{1,T} \mathrm{e}^{\lambda_1^1 t} \begin{bmatrix} 1 \\ e_1^1 \end{bmatrix} + C_2^{1,T} \mathrm{e}^{\lambda_1^2 t} \begin{bmatrix} 1 \\ e_1^2 \end{bmatrix} \tag{43}$$

with the eigenvector components $e_1^{1,2} = \frac{1}{s^2}\left(\frac{\sigma^2}{2s^2} + \rho - \lambda_1^{1,2}\right)$. Boundary conditions give us $C_1^{1,T} = \frac{(e_1^2/e_1^1)v_1(T)\mathrm{e}^{-\lambda_1^2 T} - p_1(0)/e_1^1}{(e_1^2/e_1^1)\mathrm{e}^{(\lambda_1^1-\lambda_1^2)T} - 1}$, and $C_2^{1,T} = \frac{-\mathrm{e}^{(\lambda_1^1-\lambda_1^2)T} p_1(0)/e_1^1 - \mathrm{e}^{-\lambda_1^2 T} v_1(T)}{(e_1^2/e_1^1)\mathrm{e}^{(\lambda_1^1-\lambda_1^2)T} - 1}$. Note that if $\tilde{v}(t,x) \in \mathcal{H}$ then $\lim_{t\to+\infty} |v_n(t)| < +\infty$ for all $n \geq 0$. It is also known that $|p_n(0)| < +\infty$. Since $\lambda_1^1 < 0 < \lambda_1^2$ we observe that $\mathrm{e}^{(\lambda_1^1-\lambda_1^2)T}, \mathrm{e}^{-\lambda_1^2 T} \to 0$ as $T \to +\infty$ so that in the limit, $C_1^{1,T} \to p_1(0)/e_1^1$ and $C_2^{1,T} \to 0$. Therefore we have the unique solutions $v_1(t) = (p_1(0)/e_1^1)\mathrm{e}^{\lambda_1^1 t}$ and $p_1(t) = p_1(0)\mathrm{e}^{\lambda_1^1 t}$. Therefore, if $\tilde{\mathrm{p}}(0,x) \in S_1$ so that $p_1(0) = \langle \tilde{\mathrm{p}}(0,x), H_1(x)\rangle = 0$ then $v_1(t) = 0, p_1(t) = 0$ for all $t \geq 0$.

(iii) Case $n \geq 2$: In this case, from the ODE system we have

$$\begin{bmatrix} \dot{v_n} \\ \dot{p_n} \end{bmatrix} = \begin{bmatrix} \frac{\sigma^2 n}{2s^2} + \rho & 0 \\ -\frac{n}{s^2 R} & -\frac{\sigma^2 n}{2s^2} \end{bmatrix}. \tag{44}$$

Therefore $v_n(t) = v_n(0)\mathrm{e}^{(\frac{\sigma^2 n}{2s^2}+\rho)t}$, for which the unique solution allowing $\tilde{v}(t,x) \in \mathcal{H}_G$ for all $t \geq 0$ is $v_n(t) = 0$. Therefore $p_n(t) = p_n(0)\mathrm{e}^{-\frac{nt}{s^2 R}}$ is the unique solution to the ODE on p_n.

In the preceeding discussion we have shown that the unique $\mathcal{H}_G$ solution to the perturbation system has the properties $\{v_0(t) = 0,\ v_1(t) = \frac{s^2 p_1(0)}{\frac{\sigma^2}{2s^2}+\rho-\lambda_1^1}\mathrm{e}^{\lambda_1^1 t},\ v_n(t) = 0$ for all $n \geq 2,\}$, and $\{\ p_0(t) = 0,\ p_1(t) = p_1(0)\mathrm{e}^{\lambda_1^1 t}$ and $p_n(t) = p_n(0)\mathrm{e}^{\frac{-nt}{s^2 R}}$ for all $n \geq 2\}$. Therefore using Parseval's theorem $||\tilde{\mathrm{p}}(t,x)||_{L^2(\mathrm{p}^\infty(x)\mathrm{d}x;\mathbb{R})} = \left(p_1^2(0)\mathrm{e}^{2\lambda_1^1 t} + \sum_{n=2}^{+\infty} p_n^2(0)\mathrm{e}^{-\frac{2nt}{s^2 R}}\right)^{\frac{1}{2}}$ where $\lambda_1^1 < 0$, and the Lebesgue dominated convergence theorem, we have that $\mathrm{p}_G^\infty(x)$ is linearly asymptotically stable with respect to perturbing densities in $S(\epsilon)$. □

Remark 3. *For the case $a = -\rho$, there exists a continuum of stationary solutions, similar to the models considered in ([11], [13]). Stability of mean consensus models can be proved in the case of such a continuum of solutions by imposing additional restriction of mean preserving perturbations ($\tilde{\mathrm{p}}(0,x) \in S_1$) as in [13] or via contraction mapping arguments ([1], [9], [7]). However, we do not treat this case in order to avoid such unrealistic assumptions on the density perturbation.*

We state a theorem regarding the mean consensus property [13] of the steady state MFG control law. Let us denote a finite set of agents $\mathcal{A} := \{x^i\}_{1\leq i\leq N}$, identified by their individual states x^i with individual dynamics given by equation (1). The set of agents $\mathcal{A}$ is said to have the mean consensus property if $\lim_{t\to+\infty} |\mathbb{E}[x_t^i - x_t^j]| = 0$ for any two agents $x^i, x^j \in \mathcal{A}$. The assumption below is required to prove mean consensus for a set of agents in our consensus model.

(B3) $\sup_{1\leq i\leq N} \mathbb{E}[|x_0^i|^2] < +\infty$ for the set $\mathcal{A}$.

Theorem V.6. *Let (**B1**, **B2**, **B3**) hold. Let $(v^\infty, \mathrm{p}^\infty)$ be the steady state solutions to the optimality system (5, 6) given in lemma* (V.2). *The steady state MF control law $u^\infty(x) = -\nabla v^\infty(x)/R$ applied to a set of agents $\mathcal{A}$, in the MFG model given by equations (1,2,) with $\nu(x) = \frac{1}{2}ax^2$ and consensus cost* (30) *results in a mean consensus with individual asymptotic variance $s^2 = \frac{\sigma^2}{2(a+\frac{\eta}{R})^2}$.*

The proof is a straightforward modification of that in [13], and is omitted. Since the fixed point density is unique in the generic case, there is no *initial* mean consensus, i.e. the consensus mean is independent of the initial mean of the population.

Theorems (IV.4,V.5) show that in the population model (with nonlinear agent dynamics) as well as the consensus model (with linear agent dynamics , $a \neq -\rho$), the optimal MF control law $u^*(t,x) = u^\infty(x) - \nabla\tilde{v}(t,x)/R$ for a density of agents under small S_0 perturbations is in general time-varying, and hence different from the static steady controller $u^\infty(x) = -\nabla v^\infty(x)/R$. In the next section we study the local stabilizing property of the static steady MF controller with respect to small S_0 perturbations in the steady state density, for both MFG models with nonlinear Langevin agent dynamics and general cost functions.

VI. Steady Controller: Static State Feedback

We consider the stability of a population of agents in a MFG, under the action of static state feedback provided by the steady state MFG solution. Let $(v^\infty, \mathrm{p}^\infty)$ be a fixed point for the MF system (5, 6). Consider a perturbed density of agents $\mathrm{p}^\infty(1+\epsilon\tilde{\mathrm{p}})$ as before. The static feedback MF control law $u^\infty(t,x) = -\nabla v^\infty(x)/R$ for agents governed by (1) is said to be locally *stabilizing* for a steady state density $\mathrm{p}^\infty(x)$, if the density perturbation $\tilde{\mathrm{p}}(t,x)$ governed by (11) with $\tilde{v}(t,x) \equiv 0$, decays to zero.

From equation (17), the perturbation dynamics under the static feedback are given by $\partial_t \tilde{\mathrm{p}} = \mathcal{L}\tilde{\mathrm{p}}$. Local stability therefore depends only on the eigen properties of the generator $\mathcal{L}$. Assuming (**A1**, **A2**) hold, theorems IV.1 and IV.2 imply non-negativity of spectrum of $\mathcal{L}$, which in turn yields stability w.r.t. density perturbations in S_0. Notice that this result is independent of the cost function $q(x,\mathrm{p})$. Therefore, the static feedback under the steady controller is locally stabilizing.

We demonstrate local linear stability property under decentralized static state feedback in two 1D numerical examples. We consider the example of a bistable Langevin potential, $\nu(x) = \alpha(\frac{x^4}{4} - \frac{x^2}{2})$, $\alpha > 0$, for both models considered. Open loop dynamics (1) under this potential would cause agents to *fall* into either one of the wells and exhibit a bimodal distribution at infinite time.

We use Chebfun [33] to solve for steady states of the MF system (5, 6) [7]. Monte Carlo simulations are performed for Langevin dynamics (1) using the nonlinear static feedback controller. Trajectories for $N = 500$ agents are simulated with 100 stochastic realizations each. We observe an initial distribution of agents decay to the steady state density over the total simulation time T, in both cases.

In the population model, a combined quadratic state and log density cost $q(t,x) = \frac{1}{2}Q(x-1)^2 - \ln \mathrm{p}(t,x)$ is designed. This models a population of agents with a tendency to imitate each other while moving towards the preferred state $x = 1$. Initial states of agents are sampled from a uniform density over $[-2, 2]$. We observe that for the log density cost, in Fig. 1b that some agents which are initially stuck in the potential well centered at $x = -1$ are able to escape it, to the preferential well centered at $x = 1$, given sufficient time. In figure 1c, we see that at $t = T/5$ the dynamics are dominated by the bistable potential but as time increases $t = 2T/5$, $t = T$, the density becomes unimodal with a mean close to the preferred state $x = 1$. Finally the stationary density from the PDE computation is achieved by the agents at $t = T$.

In the consensus model case, the cost (30) is used in conjunction with the long-time-average utility (7). Analytical stability results in the consensus cost case with the bistable potential, were presented by the authors in [7]. However, those results pertain to local stability of the optimal (time-varying) MFG control, in contrast with the decentralized static MF control considered here. Note that there are two steady state densities, with mean values $\mu^* = \pm 1$. We use the control law corresponding to the right well ($\mu^* = 1$). Initial states of agents are sampled from a uniform density over $[-3, 1]$. Since the initial density has a negative mean, at $t = T/5$ we notice that there are more agents in the left well. However as time increases, we see that more agents migrate into the right well under the control. At $t = T$ the PDE solution to the stationary density which is slightly bimodal, is recovered by the Monte Carlo simulation. Although we are using the consensus cost, a high control cost causes some agents to be in the well centered at $x = -1$. Most agents are seen to escape from the left well and move into the right well in figure 2a. However, due to the high noise intensity combined with low control authority, some agents are seen to move in the opposite direction as well. Finally, from stochastic means in Fig. 2c we see that unlike the linear case where mean consensus is guaranteed (theorem V.6), mean consensus is not achieved in the case with nonlinear passive dynamics.

VII. Conclusions

In this paper, we have studied MFGs for agents with multidimensional nonlinear Langevin dynamics, and provided a framework for stability analysis of fixed points in such systems. The key idea is to use the detailed balanced property of the closed-loop generator to characterize the eigenvalue spectrum of the perturbation forward-backward system, hence extending existing methods that deal with integrator agent dynamics. While we demonstrate this approach in the discounted cost case, it is also applicable to MFGs using the LTA cost functional. Using the presented approach, conditions on the stationary solutions and explicit control design constraints have been obtained for guaranteeing stability in a population distribution and a mean consensus model. We also provide a mean consensus result for the case where the Langevin potential is quadratic, with individual asymptotic variance depending on the linear drift.

It is also shown that under certain conditions on the stationary solution, the steady MF controller providing decentralized static feedback is locally stabilizing. We illustrate this fact by Monte Carlo simulations for population and consensus cost models with non-Gaussian steady state behaviour.

The most general class of (uncontrolled) diffusions which possess the detailed balance property are *reversible diffusions* with possibly multiplicative noise. Hence, the approach presented here can be extended to provide stability results for the corresponding MFG models. Generalizing our results to second order Langevin systems will be a topic of future work. Such MFG systems must be treated separately, since the concerned closed loop generator in that case is a combination of a Liouville operator and generator $\mathcal{L}$ in this paper.

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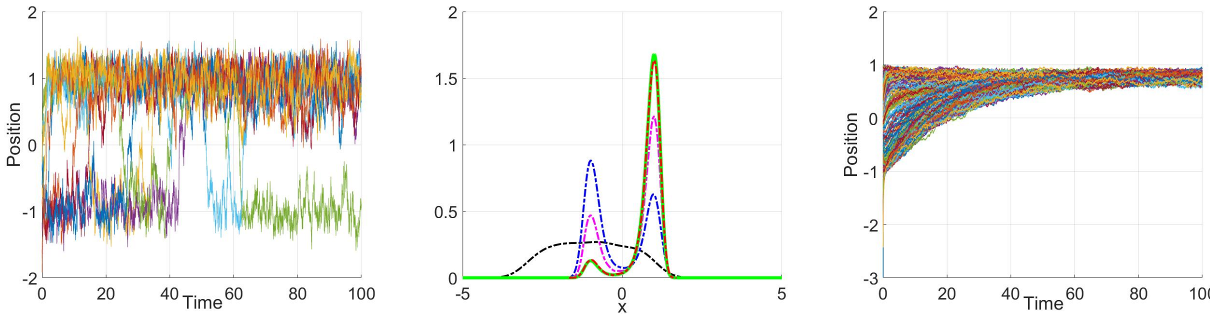

(a) (b) (c)

Fig. 2: Consensus cost model with long-time-average utility, $\alpha = 1.5$, $\sigma = 0.5$, and $R = 235$: (a) Stochastic paths for ten agents (b) Evolution of density at various times, $t = 0$ (black), $t = T/5$ (blue), $t = 2T/5$ (pink), $t = T$ (red) to the PDE solution (green) (c) Stochastic means of all agents

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