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This paper studies the control of constrained systems whose dynamics and constraints switch between a finite set of modes over time according to an exogenous input signal. We define a new type of control invariant sets for switched constrained systems, called switch-robust control invariant (switch-RCI) sets, that are robust to unknown mode switching and exploit available information on minimum dwell-time and admissible mode transitions. These switch-RCI sets are used to derive novel necessary and sufficient conditions for the existence of a control-law that guarantees constraint satisfaction in the presence of unknown mode switching with known minimum dwell-time. The switch-RCI sets are also used to design a recursively feasible model predictive controller (MPC) that enforces closed-loop constraint satisfaction for switched constrained systems. We show that our controller is non-conservative in the sense that it enforces constraints on the largest possible domain i.e. constraints can be recursively satisfied if and only if our controller is feasible. The MPC and switch-RCI sets are demonstrated on a vehicle lane-changing case study.

1 Introduction

Switched systems are a class of hybrid systems in which the dynamics switch between distinct modes over time according to a switching signal [1]. In particular, this paper considers the control of switched systems in which the switching signal is an exogenous input to the system. This often arises in applications where the dynamic mode is selected by some high-level control logic, for instance, gearshifts in vehicles [2], walking robots [3, 4], and HVAC systems with subsystem (de)activation [5, 6]. The switched systems studied in this paper are subject to mode dependent constraints on the state and input. We derive necessary and sufficient conditions for the existence of a controller that can enforce these constraints in the presence of unknown mode-switching. Furthermore, we provide an algorithm for computing the set of initial conditions where these necessary and sufficient conditions hold. Finally, we provide a recursively feasible model predictive controller that provides closed-loop constraint satisfaction.

If the mode can switch arbitrarily fast, then the switched system is equivalent to an uncertain system modeled by a differential inclusion. In particular, linear switched systems can be modeled as polytopic linear parameter varying systems [7–11]. For nonlinear switched systems with constraints, viability theory [12] can be used to characterize the set of states for which there exists a controller that enforces constraint satisfaction. However, many results from viability theory are non-constructive, whereas this paper includes a computationally efficient algorithm for computing the viability domain of the switched system and a model predictive controller for realizing constraint enforcement. Furthermore, viability theory and parameter varying systems are conservative when the mode switches are infrequent. In fact, it is well-known that switching modes can cause instability, even when each mode is stable [13]. However, if the amount of time the system dwells in each mode (called the dwell-time) is sufficiently large, then the switched system will inherit the stability of its modes [14]. This result was extended in [15] by only requiring that the average dwell-time between consecutive mode switches is sufficiently large. Thus, knowledge of the dwell-time of the switching signal can lead to less conservative stability analysis. This paper derives analogous dwell-time results for constraint satisfaction, rather than stability, in switched systems with state and input constraints.

In the first part of this paper, we characterize the set of initial conditions for which there exists a controller that can enforce constraints for all admissible switching sequences that possess a minimal dwell-time. This is accomplished using invariant sets, which are a fundamental tool for analyzing and controlling constrained systems [16]. Previous work [17] has studied robust control invariant sets for switched systems subject to external disturbances from a convex set where the mode-switching is the control input. In this paper, the roles of the control-inputs and disturbances are reversed i.e. the switching signal is the disturbance and the inputs to each mode are controlled, which obviously leads to substantially different results. The works [18,19] studied robust positive invariant sets for switched systems in which the switching signal is a disturbance, as in this paper. However, the class of switched systems studied did not include control inputs. Thus, this paper complements and extends those results [18,19] to non-autonomous switched systems.

Controlled switched systems with exogenous switching were studied in [20] where λ-contractive sets were used to show asymptotic stability of the origin for linear switched systems. Although constraints on the states and inputs were not considered, the results from [20] can be used to derive sufficient local conditions for constraint satisfaction for linear switched systems, provided the state and input constraints contain the origin in their respective interiors. However, these sufficient conditions are not necessary since convergence is not required for constraint satisfaction. Indeed, the necessary conditions derived in this paper do not even require that the nonlinear dynamics of each mode share a common stabilizable equilibrium state.

In [21], conditions for constraint satisfaction were found by backward propagating the maximal control invariant subsets of the i-th mode state constraints and the one-step backward reachable sets for each mode j in a spanning sub-graph of the allowable mode transitions. The paper [21] continued by providing computationally efficient algorithms for approximating the set of initial states where their conditions hold. A similar approach can be found in [22]. This paper contributes alternative necessary and sufficient conditions for constraint satisfaction, which are better suited for model predictive control design, as we show in the second part of this paper. Likewise, another contribution of this paper is an algorithm for computing the set of all initial states where these conditions hold that only requires computing mode-dependent reachable sets, instead of mode-dependent invariant sets. This reduces the computational burden since the sub-algorithm for computing mode-dependent invariant sets requires repeated computation of mode-dependent reachable sets and may not converge in a finite-number of iterations.

The second part of this paper studies the synthesis of model predictive controllers for constraint enforcement for switched constrained systems. Previous work [22–27], has proposed model predictive control (MPC) architectures for switched systems and then study the closed-loop properties of these controllers. In particular, [26] studied the case where the MPC is given, and derives conditions to certify its asymptotic stability and recursive feasibility as a function of the dwell-time. In contrast, this paper begins by analyzing the open-loop properties of switched systems using their invariant sets and then uses this analysis to guide our controller synthesis. This approach leads to the design of a recursively feasible MPC that enforces closed-loop constraint satisfaction on the largest possible domain i.e. constraint satisfaction is possible if and only if our controller is feasible.

This paper is organized as follows. In Section 2, we formally define switch constrained systems and the constraint enforcement problem. In Section 3, we define switch-RCI sets and present our first two main contributions; (1) necessary and sufficient conditions for constraint satisfaction, and (2) an algorithm for computing the set of initial states for which these conditions hold. The algorithm we provide for computing these switch-RCI sets is a modification of the standard invariant set algorithm, meaning that we can harness existing numerical tools [35] from traditional constrained control [16] in our algorithm. In addition, we analyze the properties of switch-RCI sets. First, we provide simple conditions for when the switch-RCI sets are compact. Second, we provide a simple linear program for checking when switch-RCI sets are non-empty for linear switched systems with polytopic constraints. Since our conditions are necessary, the feasibility of this linear program tells us when constraint satisfaction is possible.

Our third main contribution is presented in Section 4 where we use switch-RCI sets to design model predictive controllers that enforce closed-loop constraint satisfaction. We show that our controllers are recursively feasible. Furthermore, we show that our controllers are not conservative, but rather enforce constraint satisfaction on the largest possible domain. For switched constrained linear systems with linear dynamics, polytopic constraints, and a quadratic (linear) cost, our MPC controllers can be formulated as a standard quadratic (linear) program. Thus, the computational complexity of our switched-MPC is comparable to traditional MPC.

In Section 5, we apply the proposed approach to a case study in vehicle lane-changing control. We model the lane-changing vehicle as a switched constrained system in order to decouple the high-level path-planning (choosing the lane) from the low-level path following (steering the vehicle). Switch-RCI sets are used to design a path-following MPC with a mode-dependent prediction model that enforces the time-varying lane constraints specified by the high-level path-planner. The only restriction that the controller imposes on the path-planner is the minimum amount of time between lane change requests, i.e., the minimum dwell-time. Thus, switch-RCI sets could be used to simplify graph-based path-planners exploiting invariant sets [36–38].

The theoretical results presented in this paper make no assumptions about the state and input constraint sets. However, for the specific vehicle lane changing case study, we consider polytopic [28, 29], rather than ellipsoidal [30, 31], sets, for several reasons. First, polytopes naturally describe the lane boundaries and input constraints on the vehicle. Second, unlike ellipsoids [32], polytopes are closed under Minkowski addition, meaning that the reachable sets of switched linear systems with polytopic constraints, are polytopes. Indeed, the maximal switch-RCI sets are polytopes for this case study. In contrast, ellipsoidal sets can only (unsatisfactorily) approximate the system's reachable and invariant sets, whereas any convex set can be approximated with arbitrary precision by a polytope [33]. Third, polytopic sets are better suited for real-time optimization since the resulting MPC for the case study is a quadratic program [34]. In contrast, ellipsoidal sets would result in a quadratically constrained quadratic program, which is more computationally demanding and for which there are fewer options for real-time implementation.

1.0.1 Definitions

A set $\mathcal{C}$ is control invariant for the system $x^+ = f(x,u)$ if for any $x \in \mathcal{C}$, there exists $u \in \mathcal{U}$ such that $f(x,u) \in \mathcal{C}$. A necessary and sufficient condition for control invariance is $\mathcal{C} \subseteq \mathrm{Pre}(\mathcal{C})$ where $\mathrm{Pre}(\Omega) = \big\{x : \exists u, f(x,u) \in \Omega\big\}$.

2 Problem Statement

In this section, we define switched constrained systems and formulate the constraint satisfaction problem.

2.1 Switched Constrained System

This paper studies the control of constrained systems whose dynamics and constraints switch between a finite set of modes over time. We consider the following switched constrained system

$$x(t+1) = f_{\sigma(t)}\big(x(t), u(t)\big) \tag{1a}$$
$$x(t) \in \mathcal{X}_{\sigma(t)} \tag{1b}$$
$$u(t) \in \mathcal{U}_{\sigma(t)} \tag{1c}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state and $u(t) \in \mathbb{R}^{n_u^{\sigma(t)}}$ is the input. The switching sequence $\sigma : \mathbb{N} \to \mathbb{I}$ is an unknown exogenous input that switches the dynamics $f_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and the constraint sets $\mathcal{X}_i \subseteq \mathbb{R}^{n_x}$ and $\mathcal{U}_i \subseteq \mathbb{R}^{n_u^i}$ between a finite number of modes $\mathbb{I} \subset \mathbb{N}$. The number of inputs n_u^i may depend on the mode $i \in \mathbb{I}$.

Remark 1. *The autonomous switched system*

$$x(t+1) = f_{\sigma(t)}\big(x(t)\big)$$
$$x(t) \in \mathcal{X}_{\sigma(t)}$$

is a special case of the switched system (1) *with* $\mathcal{U}_i = \{0\}$ *for each mode* $i \in \mathbb{I}$. *Thus, this paper generalizes the results from [18, 19].*

2.2 Admissible Switching Sequences

The exogenous switching sequence σ acts as a disturbance on the switched system (1). In this section, we define the constraint set for this disturbance.

Two common constraints on the switching sequence are dwell-time restrictions and mode transition restrictions. The dwell-time of a mode $i \in \mathbb{I}$ is the minimal amount of time the switching sequence $\sigma : \mathbb{N} \to \mathbb{I}$ dwells in that mode

$$\mathrm{dwell}_i(\sigma) = \min\big\{\tau_{s+1} - \tau_s : \sigma(\tau_s) = i, s \in \mathbb{N}\big\}.$$

where the switching instances $\tau_s \in \mathbb{N}$ are the discrete-times at which the switching sequence changes mode $\sigma(\tau_s) \neq \sigma(\tau_s - 1)$. Restricting the dwell-times means that we only consider switching sequences σ with dwell-times of at least d_i time-instances for each mode $i \in \mathbb{I}$. Note that the unrestricted case is a special case of the constrained case with $d_i = 1$ for each mode $i \in \mathbb{I}$. The properties of the time-varying system (1) depend on the remaining amount of dwell-time which we will denote by $\delta(t)$ where

$$\delta(t+1) = \begin{cases} \max\{\delta(t)-1, 0\} & \text{if } \sigma(t+1) = \sigma(t) \\ d_{\sigma(t+1)} & \text{otherwise.} \end{cases}$$

Another common constraint on the switching sequence σ is a restriction on the admissible mode transitions. The admissible mode transitions can be specified by a directed graph $\mathbb{G} = (\mathbb{I}, \mathbb{E})$ where the graph nodes $\mathbb{I}$ are the modes of the switched system (1) and each directed edge $(i, j) \in \mathbb{E}$ indicates that a switch from mode $\sigma(\tau_s) = i$ to mode $\sigma(\tau_{s+1}) = j$ is allowed. The unrestricted case is a special case of the constrained case where the mode transition graph $\mathbb{G}$ is the complete graph.

The set of switching sequences σ that satisfy the dwell-time and mode transition restrictions is described by the following disturbance set

$$\Sigma(d, \mathbb{G}) = \big\{\sigma : \mathbb{N} \to \mathbb{I} : \text{dwell}_i(\sigma) \geq d_i, \\ (\sigma(\tau_s), \sigma(\tau_{s+1})) \in \mathbb{E},\ \forall s \in \mathbb{N}\big\}.$$

For brevity, we will often use the short-hand $\Sigma = \Sigma(d, \mathbb{G})$.

2.3 Problem Statement

The objective of this paper is to determine whether there exists a controller that guarantees constraint satisfaction for all admissible switching sequence $\sigma \in \Sigma$. This problem is formally stated below.

Problem 1. *For which initial states x_0, modes σ_0, and remaining dwell-times δ_0 does there exist a control-law $u = \kappa\big(x, \sigma, \delta\big)$ that guarantees constraint satisfaction $u(t) \in \mathcal{U}_{\sigma(t)}$ and $x(t) \in \mathcal{X}_{\sigma(t)}$ for all admissible switching sequences $\sigma \in \Sigma$ and future times $t \geq t_0$.* □

Problem 1 is analogous to the problem of stabilizing an unconstrained switched system. However, the analogy is imperfect. Two key issues specific to constraint satisfaction are highlighted below.

Constraint satisfaction is inherently a local property for switched systems (1) with non-trivial constraints $\mathcal{X}_i \subset \mathbb{R}^{n_x}$. Thus, we must consider the set of initial conditions for which it is possible to devise a control-law that guarantees constraint satisfaction. For switched systems (1), the initial conditions include, not only, the initial state $x(t_0) = x_0$, but also, the initial mode $\sigma(t_0) = \sigma_0$ and the remaining dwell-time $\delta(t_0) = \delta_0$. Thus, Problem 1 was formulated in terms of the finding the initial conditions for which constraint satisfaction can be guaranteed.

The unknown nature of the switching sequence plays an important role in constraint satisfaction. It may be possible to satisfy constraints for every admissible switching sequence Σ provided that the future switching sequence is known, but impossible otherwise. Conversely, it may be possible to satisfy constraints using only knowledge of the current and past modes, but impossible otherwise. Thus, Problem 1 was formulated in terms of the existence of a (feedback) control-law that intrinsically only has knowledge of the current mode $\sigma(t) \in \mathbb{I}$ rather than the existence of constraint satisfying trajectories.

3 Switch-Robust Control Invariant Sets

In this section, we define a new class of control invariant sets for switched constrained systems (1) that are robust to unknown mode switching $\sigma \in \Sigma$. These switch-robust control invariant (switch-RCI) sets are used to solve Problem 1.

3.1 Definition of Switch-RCI Sets

Switch-RCI sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ are constraint admissible control invariant sets that are robust to mode switches. In order to satisfy constraints for any possible switching sequence $\sigma \in \Sigma$, we must satisfy constraints for the constant sequence $\sigma(t) = i$ for all $t \geq t_0 \in \mathbb{N}$. Thus, each switch-RCI set $\mathcal{C}_i$ must be a constraint admissible control invariant set for its mode $i \in \mathbb{I}$. Furthermore, when an unexpected mode change occurs $i \to j$, it must be possible to satisfy the constraints of the new mode $j \in \mathbb{I}$. This can be ensured by requiring that each set $\mathcal{C}_j$ is reachable from $\mathcal{C}_i$ under the dynamics and constraints of mode $j \in \mathbb{I}$ within the dwell-time d_j of that mode for $(i,j) \in \mathbb{E}$. This can be quantified using the predecessor-operator, defined recursively by

$$\mathrm{Pre}_i^0(\Omega) = \Omega \tag{2a}$$

$$\mathrm{Pre}_i^{k+1}(\Omega) = \big\{x \in \mathcal{X}_i : \exists u \in \mathcal{U}_i \text{ s.t. } f_i(x,u) \in \mathrm{Pre}_i^k(\Omega)\big\} \tag{2b}$$

for $k \in \mathbb{N}$. The set $\mathrm{Pre}_i^k(\Omega) \subseteq \mathcal{X}_i$ is the set of states $x \in \mathcal{X}_i$ that can be mapped, under the dynamics of mode $i \in \mathbb{I}$, into the set Ω in k discrete-time instances without violating the state $\mathcal{X}_i$ and input $\mathcal{U}_i$ constraints of the mode $i \in \mathbb{I}$. Switch-RCI sets are defined formally below.

Definition 1 (Switch-RCI Sets)**.** *A collection of sets $\mathcal{C}_i \subseteq \mathcal{X}_i$ for $i \in \mathbb{I}$ is called switch-robust control invariant if they are control invariant $\mathcal{C}_i \subseteq \mathrm{Pre}_i^1(\mathcal{C}_i)$ for each mode $i \in \mathbb{I}$ and mutually reachable $\mathcal{C}_i \subseteq \mathrm{Pre}_j^{d_j}(\mathcal{C}_j) \subseteq \mathcal{X}_j$ within the dwell-time d_j for each admissible mode transition $(i,j) \in \mathbb{E}$.*

Switch-RCI sets can be used to formulate necessary and sufficient conditions for Problem 1. The following lemma uses switch-RCI sets to provide a sufficient condition for guaranteeing constraint satisfaction.

Lemma 1. *If the initial state $x(t_0)$, mode $\sigma(t_0)$, and remaining dwell-time $\delta_0(t_0)$ are contained in the set*

$$\mathcal{IC} = \big\{(x_0, \sigma_0, \delta_0) : x_0 \in \mathrm{Pre}_{\sigma_0}^{\delta_0}\big(\mathcal{C}_{\sigma_0}\big)\big\} \tag{3}$$

then there exists a control-law $\kappa(x,\sigma,\delta)$ that guarantees constraint satisfaction for all future times $t \geq t_0$ and all admissible switching sequences $\sigma \in \Sigma$.

Proof. We will prove the existence of a control-law $\kappa(x,\sigma,\delta)$ by showing that at each time $t \geq t_0$ there exists a feasible control input $u(t) \in \mathcal{U}_{\sigma(t)}$ that satisfies state constraints $x(t+1) \in \mathcal{X}_{\sigma(t+1)}$ for any admissible switching sequence $\sigma \in \Sigma$.

First, we show that if $x(\tau_s) \in \mathrm{Pre}_{\sigma(\tau_s)}^{d_{\sigma(\tau_s)}}\big(\mathcal{C}_{\sigma(\tau_s)}\big)$ at the s-th switching instance $\tau_s \geq t_0$ then it is possible to satisfy the state and input constraints until the next switching time $\tau_{s+1} \geq \tau_s$. By the definition (2) of $\mathrm{Pre}_{\sigma(\tau_s)}^{d_{\sigma(\tau_s)}}\big(\mathcal{C}_{\sigma(\tau_s)}\big)$ there exists $u(t) \in \mathcal{U}_{\sigma(t)}$ such that $x(t+1) \in \mathcal{X}_{\sigma(t)}$ for $t = \tau_s, \dots, \tau_{s+1} + d_{\sigma(\tau_s)} - 1$ and $x(\tau_s + d_{\sigma(\tau_s)}) \in \mathcal{C}_{\sigma(\tau_s)}$ since $\sigma(t) = \sigma(\tau_s)$. For $t = \tau_s + d_{\sigma(\tau_s)}, \dots, \tau_{s+1} - 1$, there exists $u(t) \in \mathcal{U}_{\sigma(t)}$ such that $x(t+1) \in \mathcal{C}_{\sigma(\tau_s)} \subseteq \mathcal{X}_{\sigma(t)}$ since $\mathcal{C}_{\sigma(t)} = \mathcal{C}_{\sigma(\tau_s)}$ is a constraint admissible control invariant set for the dynamics and constraints of mode $\sigma(\tau_s) \in \mathbb{I}$ where $\tau_{s+1} \geq \tau_s + d_{\sigma(\tau_1)}$. Thus, the input and state constraints are satisfied between switching times $t \in [\tau_s, \tau_{s+1} - 1]$ for $x(\tau_s) \in \mathrm{Pre}_{\sigma(\tau)}^{d_{\sigma(\tau)}}\big(\mathcal{C}_{\sigma(\tau)}\big)$.

Next, we show by induction that $x(\tau_s) \in \mathrm{Pre}_{\sigma(\tau_s)}^{d_{\sigma(\tau_s)}}(\mathcal{C}_{\sigma(\tau_s)})$ holds at every switching time τ_s for $s \in \mathbb{N}$. Since $x(t_0) \in \mathrm{Pre}_{\sigma_0}^{\delta_0}\big(\mathcal{C}_{\sigma_0}\big)$ and $\mathcal{C}_{\sigma_0}$ is control invariant there exists a control sequence $u(t) \in \mathcal{U}_{\sigma_0}$ such that $x(\tau_1) \in \mathcal{C}_{\sigma_0} \subseteq \mathrm{Pre}_{\sigma(\tau_1)}^{d_{\sigma(\tau_1)}}(\mathcal{C}_{\sigma(\tau_1)})$ at the first switching time $\tau_1 \geq t_0 + \delta_0$. If $x(\tau_s) \in \mathrm{Pre}_{\sigma(\tau_s)}^{d_{\sigma(\tau_s)}}(\mathcal{C}_{\sigma(\tau_s)})$ then the previously defined input $u(t)$ yields $x(\tau_{s+1}) \in \mathcal{C}_{\sigma(\tau_s)} \subseteq \mathrm{Pre}_{\sigma(\tau_{s+1})}^{d_{\sigma(\tau_{s+1})}}(\mathcal{C}_{\sigma(\tau_{s+1})})$ at time τ_{s+1}. Thus, we conclude that it is possible to satisfy constraints for all $t \geq t_0$. $\square$

Lemma 1 says that if the switched system (1) is initialized in the set (3) then it is possible to design a controller that satisfies constraints. Switch-RCI sets can also be used to find a necessary condition for guaranteeing constraint satisfaction.

Lemma 2. *If there exists a control-law $\kappa(x,\sigma,\delta)$ that guarantees constraint satisfaction $\kappa(x(t),\sigma(t),\delta(t)) \in \mathcal{U}_{\sigma(t)}$ and $x(t) \in \mathcal{X}_{\sigma(t)}$ for all time $t \in \mathbb{N}$ and all admissible switching sequences $\sigma \in \Sigma$, then the initial state $x(t_0)$, mode $\sigma(t_0)$, and remaining dwell-time $\delta(t_0)$ must be contained in the initial conditions set (3) for some collection of switch-RCI sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$.*

Proof. For a particular switching sequence $\sigma \in \Sigma$, let $\{x^\sigma(t)\}_{t=t_0}^\infty$ denote the resulting state trajectory produced by the switched system (1) in closed-loop with the controller $\kappa(x, \sigma, \delta)$. Let $\mathcal{T}_i^\sigma$ denote the set of states visited by $\{x^\sigma(t)\}_{t=t_0}^\infty$ while in mode $i = \sigma(t)$ after the dwell-time $\delta(t) = 0$ has expired,

$$\mathcal{T}_i^\sigma = \big\{x(t) : \sigma(t) = i, \delta(t) = 0\big\} \subseteq \mathcal{X}_i.$$

We will show that the sets $\mathcal{C}_i = \bigcup_{\sigma\in\Sigma} \mathcal{T}_i^\sigma$ are switch-RCI sets. First we show that the set $\mathcal{C}_i = \bigcup_{\sigma\in\Sigma} \mathcal{T}_i^\sigma$ are control invariant. Let $x \in \mathcal{C}_i$. Then there exists a switching sequence $\sigma_1 \in \Sigma$ and time $t_1 \in \mathbb{N}$ such that $x = x^{\sigma_1}(t_1)$. Furthermore, there exists a second switching sequence $\sigma_2 \in \Sigma$ such that $\sigma_1(t) = \sigma_2(t)$ for $t = t_0, \dots, t_1$ and $\sigma_2(t_1 + 1) = i$. Thus, there exists $u \in \mathcal{U}_i$ such that $f_i(x, u) \in \mathcal{C}_i \supseteq \mathcal{T}_i^{\sigma_2}$ by definition of $\mathcal{T}_i^{\sigma_2}$.

Next we show that the sets $\mathcal{C}_i = \bigcup_{\sigma\in\Sigma} \mathcal{T}_i^\sigma$ are mutually reachable. As before, for each $x \in \mathcal{C}_i$ there exists $\sigma_1 \in \Sigma$ and $t_1 \in \mathbb{N}$ such that $x = x^{\sigma_1}(t_1)$. Furthermore, there exists a third switching sequence $\sigma_3 \in \Sigma$ such that $\sigma_1(t) = \sigma_3(t)$ for $t = t_0, \dots, t_1$ and $\sigma_3(t) = j$ for $t = t_1 + 1, \dots, t_1 + d_j$ with $x(t_1+d_j) \in \mathcal{C}_j \supseteq \mathcal{T}_j^{\sigma_3}$ by construction. Thus, $\mathcal{C}_i \subseteq \mathrm{Pre}_j^{d_j}\big(\mathcal{C}_j\big)$ and therefore the collection $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ is switch-RCI by Definition 1.

Finally, we note that $x_0 \in \mathrm{Pre}_{\sigma_0}^{\delta_0}\big(\mathcal{C}_{\sigma_0}\big)$ by the definition of the set $\mathcal{C}_{\sigma_0} = \bigcup_{\sigma\in\Sigma} \mathcal{T}_{\sigma_0}^\sigma$. □

Lemma 2 says in order to satisfy constraints, the switched system (1) must be initialized in the set (3) for some collection of switch-RCI sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$. A consequence of Lemma 2 is that it is impossible to guarantee constraint satisfaction for all possible switching sequences if the system is allowed to switch $(i, j) \in \mathbb{E}$ between modes with disjoint constraints $\mathcal{X}_i \cap \mathcal{X}_j = \varnothing$ where $\mathcal{C}_i \subseteq \mathcal{X}_i \cap \mathcal{X}_j$ by Definition 1.

The sufficient and necessary conditions of Lemma 1 and 2 can be combined using *maximal* switch-RCI sets, defined below.

Definition 2 (maximal Switch-RCI Sets)**.** *A collection of sets $\{\mathcal{C}_i^\infty\}_{i\in\mathbb{I}}$ are maximal switch-RCI if they are switch-RCI and for any collection of switch-RCI sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ we have $\mathcal{C}_i \subseteq \mathcal{C}_i^\infty$ for each mode $i \in \mathbb{I}$.*

Theorem 1. *There exists a control-law $\kappa(x, \sigma, \delta)$ that guarantees constraint satisfaction for all time $t \in \mathbb{N}$ and all admissible switching sequences $\sigma \in \Sigma$ if and only if the initial state $x(t_0)$, mode $\sigma(t_0)$, and remaining dwell-time $\delta(t_0)$ are contained in the initial conditions set* (3) *for the maximal switch-RCI sets $\{\mathcal{C}_i^\infty\}_{i\in\mathbb{I}}$.*

Theorem 1 follows directly from Definition 2 and Lemmas 1 and 2. It provides necessary and sufficient conditions for guaranteeing constraint satisfaction for switched systems. However, Theorem 1 tacitly assumes that it is possible for a collection of maximal switch-RCI sets to exist. The existence and uniqueness of these sets is discussed in the following remark.

Remark 2. *From Definition 2, it is clear that if the maximal switch-RCI sets $\{\mathcal{C}_i^\infty\}_{i\in\mathbb{I}}$ exist then they are unique. However, it is not immediately clear that it is possible for a collection of sets $\{\mathcal{C}_i^\infty\}_{i\in\mathbb{I}}$ to satisfy Definition 2. To elaborate this point, consider posing an optimization problem that maximizes the volume of the sets $\mathcal{C}_i$ for $i \in \mathbb{I}$ subject to the constraint that the collection $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ is switch-RCI. The maximal switch-RCI sets $\{\mathcal{C}_i^\infty\}_{i\in\mathbb{I}}$ would be the optimal solution of this optimization problem (upto zero-measure sets). However, the described optimization problem is a multi-objective optimization problem. In general, multi-objective optimization problems do not have optimal solutions, but rather a set of Pareto optimal solutions. Thus, maximal switch-RCI sets $\{\mathcal{C}_i^\infty\}_{i\in\mathbb{I}}$ can only exist if the Pareto set is a singleton.* □

Fortunately, the existence of sets that satisfy Definition 2 will be verified in the next section when we present an algorithm for computing maximal switch-RCI sets.

3.2 Computing Maximal Switch-RCI Sets

A collection of maximal switch-RCI sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ can be computed using Algorithm 1, which is based on the standard control invariant set algorithm [16]. Algorithm 1 initializes the estimates $\{\Omega_i^k\}_{i\in\mathbb{I}}$ of the switch-RCI sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ with the outer-approximations $\Omega_i^0 = \mathcal{X}_i \supseteq \mathcal{C}_i$ for each mode $i \in \mathbb{I}$. During each iteration, Algorithm 1 refines the outer estimates $\{\Omega_i^k\}_{i\in\mathbb{I}}$ by removing states $x \in \Omega_i^k$ that cannot be kept in the set Ω_i^k under the dynamics of mode $i \in \mathbb{I}$ and cannot reach Ω_j^k in d_j time instances under the dynamics of mode $j \in \mathbb{I}$. This is accomplished by intersecting the sets Ω_i^k with the predecessor sets $\mathrm{Pre}_i^1(\Omega_i^k)$ and $\mathrm{Pre}_j^{d_j}(\Omega_j^k)$

Algorithm 1 Maximal Switch-RCI Sets

for each mode $i \in \mathbb{I}$ **do**
  $\Omega_i^0 = \mathcal{X}_i$
end for
repeat
  for each mode $i \in \mathbb{I}$ **do**
    Update sets

$$\Omega_i^{k+1} = \Omega_i^k \bigcap \mathrm{Pre}_i(\Omega_i^k) \bigcap_{(i,j)\in\mathbb{E}} \mathrm{Pre}_j^{d_j}(\Omega_j^k) \tag{4}$$

  end for
until $\Omega_i^{k+1} = \Omega_i^k$ for all $i \in \mathbb{I}$
$\mathcal{C}_i^\infty = \Omega_i^k$ for all $i \in \mathbb{I}$.

where the predecessor-operator was defined in (2). The algorithm terminates when the estimates Ω_i^k of the switch-RCI sets $\mathcal{C}_i$ have converged $\Omega_i^{k+1} = \Omega_i^k$ for each mode $i \in \mathbb{I}$.

Algorithm 1 modifies the update rule $\Omega_i^{k+1} = \Omega_i^k \cap \mathrm{Pre}_i(\Omega_i^k)$ used to compute standard control invariant sets [16]. The estimates Ω_i^k of the switch-RCI sets are updated by intersecting the update rule for traditional control invariant sets with each set of states, $\mathrm{Pre}_j^{d_j}(\Omega_j^k)$, that can reach the invariant sets Ω_j^k of mode $j \in \mathbb{I}$. This reflects the fact that the switch-RCI sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ must be more conservative than standard control invariant sets since the switching sequence $\sigma \in \Sigma$ is a disturbance that can cause constraint violations.

The following theorem shows that the sets $\{\mathcal{C}_i^\infty\}_{i\in\mathbb{I}}$ produced by Algorithm 1 are maximal switch-RCI sets.

Theorem 2. *Algorithm 1 produces the maximal switch-RCI sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ for the system* (1).

Proof. First, we prove that the sets $\mathcal{C}_i^\infty = \lim_{k\to\infty} \Omega_i^k$ exist. Note that $\mathcal{C}_i = \varnothing$ for each $i \in \mathbb{I}$ trivial satisfies Definition 1. Thus, each of the sequences of sets $\{\Omega_i^k\}_{k\in\mathbb{N}}$ for $i \in \mathbb{I}$ are bounded from below $\Omega_i^k \supseteq \varnothing$ and are non-increasing $\Omega_i^{k+1} \subseteq \Omega_i^k$. Thus, the limits $\mathcal{C}_i = \lim_{k\to\infty} \Omega_i^k$ exist for each $i \in \mathbb{I}$.

Next, we prove that the sets $\{\mathcal{C}_i^\infty\}_{i\in\mathbb{I}}$ are switch-RCI. By definition $\mathcal{C}_i^\infty = \lim_{k\to\infty} \Omega_i^k$, the sets $\mathcal{C}_i^\infty$ are fixed-points of the update-rule (4) i.e.

$$\mathcal{C}_i^\infty = \mathcal{C}_i^\infty \bigcap \mathrm{Pre}_i(\mathcal{C}_i^\infty) \bigcap_{(i,j)\in\mathbb{E}} \mathrm{Pre}_j^{d_j}(\mathcal{C}_j^\infty).$$

Thus, each set $\mathcal{C}_i^\infty$ is control invariant since $\mathcal{C}_i^\infty \subseteq \mathrm{Pre}(\mathcal{C}_i^\infty)$. Furthermore, for each $(i,j) \in \mathbb{E}$ the set $\mathcal{C}_j^\infty$ is reachable from $\mathcal{C}_i^\infty$ in d_j time-instances since $\mathcal{C}_i^\infty \subseteq \mathrm{Pre}_j^{d_j}(\mathcal{C}_j^\infty)$. Thus, by Definition 1 the collection $\{\mathcal{C}_i^\infty\}_{i\in\mathbb{I}}$ are switch-RCI sets.

Finally, we prove that the switch-RCI sets $\{\mathcal{C}_i^\infty\}_{i\in\mathbb{I}}$ are maximal by showing that constraint satisfaction requires the initial conditions lie inside the set (3) with these sets. Suppose $x(t_0) = x_0 \notin \mathrm{Pre}_{\sigma_0}^{\delta_0}(\Omega_{\sigma_0}^k)$ for some $k \in \mathbb{N}$. We will use this fact to construct an admissible switching sequence $\sigma \in \Sigma$ that leads to a constraint violation. By the definition (2) of $\mathrm{Pre}_{\sigma_0}^{\delta_0}(\Omega_{\sigma_0}^k)$ we have $x(t_1) \notin \Omega_{\sigma(t_1)}^k$ at $t_1 = t_0 + \delta$ for any feasible control input sequence $u(t) \in \mathcal{U}_{\sigma(t)}$. By DeMorgan's law and the update-rule (4), one of the following three (non-exclusive) conditions must hold

1. $x(t_\ell) \notin \Omega_{\sigma(t_\ell)}^{k-1}$
2. $x(t_\ell) \notin \mathrm{Pre}(\Omega_{\sigma(t_\ell)}^{k-1})$
3. $x(t_\ell) \notin \mathrm{Pre}_j^{d_j}(\Omega_j^{k-1})$ for some $(i,j) \in \mathbb{E}$.

In each case, we can construct a admissible switching sequence such that $x(t_{\ell+1}) \notin \Omega_{\sigma(t_{\ell+1})}^{k-1}$ for any feasible input sequences $u(t) \in \mathcal{U}_{\sigma(t)}$ for $t = t_\ell + 1, \ldots, t_{\ell+1}$. The switching sequence $\sigma(t)$ and time $t_{\ell+1}$ for each case are given by

1. $t_{\ell+1}=t_\ell$
2. $t_{\ell+1}=t_\ell+1$ and $\sigma(t_{\ell+1}) = \sigma(t_\ell)$
3. $t_{\ell+1}=t_\ell+d_j$ and $\sigma(t) = j$ for $t = t_\ell+1,\dots,t_{\ell+1}$

By induction on $k \in \mathbb{N}$, this means that for any feasible control-law there exists an admissible switching sequence $\sigma \in \Sigma$ that produces a state constraint violation $x(t_k) \notin \Omega^0_{\sigma(t_k)}=\mathcal{X}_{\sigma(t_k)}$ at some finite time $t_k \in \mathbb{N}$. Therefore, we have proven: if $x(t_0)=x_0 \notin \mathrm{Pre}^{\delta_0}_{\sigma_0}(\Omega^k_{\sigma_0})$ for some $k \in \mathbb{N}$ then for all feasible control-laws there exists a switching sequences $\sigma \in \Sigma$ and time $t_k \in \mathbb{N}$ such that $x(t_k) \notin \mathcal{X}_{\sigma(t_k)}$. The contra-positive of the previous statement is: if there exists a feasible control-law such that for all times $t \geq t_0 \in \mathbb{N}$ and switching sequences $\sigma \in \Sigma$ we have $x(t) \in \mathcal{X}_{\sigma(t)}$ then $x(t_0+\delta_0) \in \Omega^k_{\sigma_0}$ for all $k \in \mathbb{N}$. Since $\mathcal{C}^\infty_i = \bigcap_{k\in\mathbb{N}} \Omega^k_i$, the previous statement proves that it is possible to satisfy constraints only if $x_0 \in \mathrm{Pre}^{\delta_0}_{\sigma_0}(\mathcal{C}_{\sigma_0})$. Thus, $\mathcal{C}^\infty_i$ are the maximal switch-RCI sets. □

Together, Theorems 1 and 2 mean that it is possible to satisfy constraints for all switching sequences $\sigma \in \Sigma$ if and only if the system (1) is initialized in the set (3) with the maximal switch-RCI $\{C^\infty_i\}_{i\in\mathbb{I}}$ provided by Algorithm 1. Furthermore, Theorem 2 resolves the issue described in Remark 2: A switched system (1) always has a collection of sets $\{\mathcal{C}^\infty_i\}_{i\in\mathbb{I}}$ that satisfy Definition 2 namely the (unique) limit of the sequence of sets $\{\Omega^k_i\}_{k\in\mathbb{N}}$. Of course, this collection can be empty (i.e. $\mathcal{C}^\infty_i = \varnothing$ for all $i \in \mathbb{I}$) which means that the constraints on the dwell-time and mode transitions are not restrictive enough to guarantee constraint satisfaction for all admissible switching sequences $\sigma \in \Sigma(d,\mathbb{G})$. A sufficient condition for the switched system (1) to have non-empty switch-RCI sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ is given by the following corollary.

Corollary 1. *If the system* (1) *has a collection* $\{\bar{x}_i\}_{i\in\mathbb{I}}$ *of mutually reachable* $\bar{x}_i \in \mathrm{Pre}^{d_j}_j(\{\bar{x}_j\})$ *and feasible equilibrium state* $\bar{x} = f_i(\bar{x},\bar{u}_i) \in \mathcal{X}_i$ *with* $\bar{u}_i \in \mathcal{U}_i$ *for each mode* $i \in \mathbb{I}$ *then it has non-empty switch-RCI sets* $\{\mathcal{C}^\infty_i\}_{i\in\mathbb{I}}$.

Proof. The sets $\mathcal{C}_i = \{\bar{x}_i\}$ for each $i \in \mathbb{I}$ are switch-RCI since they are control invariant and mutually reachable. □

If the switched constrained system (1) has linear dynamics and polytopic constraints then the conditions of Corollary 1 can be posed as the following linear feasibility problem

$$\begin{aligned} \min\ & \varnothing \\ \text{s.t.}\ & \bar{x}_i = A_i\bar{x}_i + B_i\bar{u}_i, \bar{x}_i \in \mathcal{X}_i, \bar{u}_i \in \mathcal{U}_i\ \forall i \in \mathbb{I} && \text{(5a)} \\ & x_{k+1} = A_j x_k + B_j u_k\ k = 0,\dots,d_j \\ & \quad x_0 = \bar{x}_i, x_{d_j} = \bar{x}_j, x_k \in \mathcal{X}_j, u_k \in \mathcal{U}_j\ \forall (i,j) \in \mathbb{E} && \text{(5b)} \end{aligned}$$

where (5a) ensures that the sets $\mathcal{C}_i = \{\bar{x}_i\}$ are control invariant and (5b) ensures that the sets are mutually reachable. Thus, the sets $\mathcal{C}_i = \{\bar{x}_i\}$ satisfy Defintion 1.

The following lemma provides a simple condition that results in compact switch-RCI sets.

Lemma 3. *Suppose the dynamics* (1a) *are continuous and the constraint sets* (1b) *and* (1c) *are compact. Then the maximal switch-RCI sets* $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ *produced by Algorithm 1 are compact.*

Proof. By the closed set condition, the predecessor-operator (2) maps closed sets to closed sets since the dynamics (1a) are continuous and the constraint sets (1b) and (1c) are compact. Thus, by the update rule (4), the set Ω^{k+1}_i will be compact if the set Ω^k_i is compact since Ω^{k+1}_i is the intersection of closed sets $\mathrm{Pre}_i(\Omega^k_i)$ and $\mathrm{Pre}^{d_j}_j(\Omega^k_j)$ with a compact set Ω^k_i. Therefore, by induction on k, each of the sets Ω^k_i is compact since $\Omega^0_i = \mathcal{X}_i$ is compact. Finally, we note that the sets $\mathcal{C}^\infty_i = \bigcap_{k=0}^\infty \Omega^k_i$ are compact since they are the (arbitrary) intersection of compact sets. □

Lemma 3 will be important in Section 4 when we use switch-RCI to design model predictive controllers for switched systems (1). The compactness of the switch-RCI sets will ensure that the infimum is achieved in the optimization problems solved by the MPC.

4 MPC for Switched Systems

The switch-robust control invariant sets $\{\mathcal{C}_i\}_{i\in\mathbb{I}}$ guarantee the existence of a control-law that satisfies constraints. However, the problem of constructing such a controller remains. In this section, we present model predictive controllers that use the switch-RCI sets $\mathcal{C}_i$ to generate control inputs that satisfy constraints.

4.1 Long-Horizon MPC

In this section, we consider an MPC where the prediction horizon $N \geq d_i$ is longer than the dwell-time of each mode $i \in \mathbb{I}$. The MPC computes the control input $u(t)$ by solving the following constrained finite-time optimal control problem

$$\begin{aligned}
\min \quad & p_{\sigma_{0|t}}(x_{N|t}) + \textstyle\sum_{k=0}^{N-1} q_{\sigma_{0|t}}(x_{k|t}, u_{k|t}) && \text{(6a)}\\
\text{s.t.} \quad & x_{k+1|t} = f_{\sigma_{0|t}}(x_{k|t}, u_{k|t}) && \text{(6b)}\\
& x_{k+1|t} \in \mathcal{X}_{\sigma_{0|t}},\ u_{k|t} \in \mathcal{U}_{\sigma_{0|t}} && \text{(6c)}\\
& x_{k+1|t} \in \mathcal{T}_{\sigma_{0|t}} \text{ for } k \geq \delta_{0|t} && \text{(6d)}
\end{aligned}$$

where $x_{0|t} = x(t)$ is the current state of the system (1), $x_{k|t}$ is the predicted state of the system under the control actions $u_{k|t}$ over the prediction horizon $N \geq d_i$, $\sigma_{0|t} = \sigma(t)$ is the current mode of the system, $\mathcal{T}_{\sigma_{0|t}}$ is the terminal constraint set, and $\delta_{0|t} = \delta(t)$ the remaining dwell-time. Since this paper is focused on constraint satisfaction, the terminal $p_{\sigma_{0|t}}(\cdot)$ and stage $q_{\sigma_{0|t}}(\cdot,\cdot)$ costs are unrestricted and can be selected to satisfy secondary control objectives such as stability or reference tracking for the individual modes. The optimal control problem (6) is solved assuming that the mode $\sigma(t) \in \mathbb{I}$ is constant $\sigma_{k|t} = \sigma_{0|t}$ over the entire horizon $k = 0, \dots, N$.

The "terminal" constraint (6d) is applied for every time instance $k \geq \delta_{0|t}$ after the remaining dwell-time $\delta_{0|t}$ has expired. This ensures that the predicted state $x_{k|t}$ enters the terminal region $\mathcal{T}_{\sigma_{0|t}}$ by the time the dwell-time expires and remains there until the next mode switch. The terminal constraint (6d) naturally loosens the constraints immediately after a mode switch and tightens the constraints as the dwell-time dwindles.

The control input is the first element $u^\star_{0|t}$ of the optimal open-loop input sequence $u^\star_{0|t}, \dots, u^\star_{N_t-1|t}$

$$u(t) = u^\star_{0|t}\big(x(t), \sigma(t), \delta(t)\big). \tag{7}$$

In contrast to traditional MPC, the controller (7) is mode-varying and time-varying since $u(t)$ depends on the current state $x(t)$, mode $\sigma(t)$, and the remaining dwell-time $\delta(t)$.

The following shows that the optimal control problem (6) is recursively feasible.

Theorem 3. *Let $\mathcal{T}_{\sigma_{0|t}} = \mathcal{C}_{\sigma_{0|t}}$ be compact switch-RCI sets. If (6) has a solution for $x(t)$ then it has a solution for $x(t+1) = f_{\sigma(t)}\big(x(t), u(t)\big)$ where $u(t) = u^\star_{0|t}$.*

Proof. Since the optimal control problem (6) has a solution at time $t \in \mathbb{N}$, there exists a feasible input sequence $u^\star_{0|t}, \dots, u^\star_{N-1|t} \in \mathcal{U}_{\sigma_{0|t}}$ that generates a feasible state trajectory $x^\star_{0|t}, \dots, x^\star_{N_t|t} \in \mathcal{X}_{\sigma_{0|t}}$ so that the terminal constraints $x^\star_{k|t} \in \mathcal{C}_{\sigma_{0|t}}$ for $k \geq \delta_{0|t}$. We will use this solution to construct a feasible solution at time $t+1 \in \mathbb{N}$ where $x_{0|t+1} = f_{\sigma_{0|t}}\big(x^\star_{0|t}, u_{0|t}\big) = x^\star_{1|t}$. We consider two cases: when a mode switch did occur $\sigma_{0|t+1} = \sigma_{0|t}$ and when one did not occur $\sigma_{0|t+1} \neq \sigma_{0|t}$.

In the first case, the input sequence $u_{k|t+1} = u^\star_{k+1|t} \in \mathcal{U}_{\sigma_{0|t}} = \mathcal{U}_{\sigma_{0|t+1}}$ for $k = 0, \dots, N-2$ and the state sequence $x_{k|t+1} = x^\star_{k+1|t} \in \mathcal{X}_{\sigma_{0|t}} = \mathcal{X}_{\sigma_{0|t+1}}$ for $k = 0, \dots, N-1$ are feasible. Furthermore, $x_{k|t+1} = x^\star_{k+1|t} \in \mathcal{C}_{\sigma_{0|t}} = \mathcal{C}_{\sigma_{0|t+1}}$ for $k \geq \delta_{0|t}$. Since $x_{N-1|t+1} = x^\star_{N|t} \in \mathcal{C}_{\sigma_{0|t}}$ lies in the control invariant set $\mathcal{C}_{\sigma_{0|t+1}} = \mathcal{C}_{\sigma_{0|t}}$, there exists a feasible input $u_{N-1|t+1} \in \mathcal{U}_{\sigma_{0|t+1}}$ such that

$$x_{N|t+1} = f_{\sigma_{0|t+1}}(x_{N-1|t+1}, u_{N-1|t+1}) \in \mathcal{C}_{\sigma_{0|t+1}}$$

where $\mathcal{C}_{\sigma_{0|t+1}} \subseteq \mathcal{X}_{\sigma_{0|t+1}}$. Thus, (6) has a feasible solution at time $t+1$ in this case.

In the second case, the fact that the mode changed $\sigma_{0|t+1} \neq \sigma_{0|t}$ means that the dwell-time must have expired before the previous time index i.e. $\delta(t) = 0$. Thus, $x_{0|t+1} = x^\star_{1|t} \in \mathcal{C}_{\sigma_{0|t}}$. Since $\mathcal{C}_{\sigma_{0|t}}$ is switch-RCI, we have $\mathcal{C}_{\sigma_{0|t}} \subseteq \mathrm{Pre}^d_{\sigma_{0|t+1}}(\mathcal{C}_{\sigma_{0|t+1}})$ where $(\sigma_{0|t}, \sigma_{0|t+1}) \in \mathbb{E}$. By the definition of the predecessor-operator (2), there exists a sequence of feasible control inputs $u_{k|t+1} \in \mathcal{U}_{\sigma_{0|t+1}}$ that produces a feasible sequence of states $x_{k+1|t+1} = f_{\sigma_{0|t+1}}\big(x_{k|t+1}, u_{k|t+1}\big) \in \mathcal{X}_{\sigma_{0|t+1}}$ for $k = 0, \dots, d_{\sigma(t+1)}$ and with $x_{d|t} \in \mathcal{C}_{\sigma_{0|t+1}}$. For $k \geq \delta(t+1) = d_{\sigma(t+1)}$ there exist feasible inputs $u_{k|t+1} \in \mathcal{U}_{\sigma_{0|t+1}}$ that produce states $x_{k+1|t+1} = f_{\sigma_{0|t+1}}\big(x_{k|t+1}, u_{k|t+1}\big) \in \mathcal{C}_{\sigma_{0|t+1}}$ since $\mathcal{C}_{\sigma_{0|t+1}}$ is a control invariant set for mode $\sigma_{0|t+1}$. Thus, (6) has a feasible solution at time $t+1$ in both cases. □

Since the optimal control problem (6) explicitly requires that the input $u(t) = u^\star_{0|t}$ and state $x(t+1) = x^\star_{1|t}$ satisfy constraints (6c), Theorem 3 means that the model predictive controller (7) guarantees constraint satisfaction for any initial state $x(t)$, mode $\sigma(t)$, and remaining dwell-time $\delta(t)$ in the domain of the controller. The following corollary characterizes the domain of the MPC (7).

Corollary 2. *Let $\mathcal{T}_{\sigma_{0|t}} = \mathcal{C}_{\sigma_{0|t}}$ be compact switch-RCI sets. Then the MPC* (7) *is defined for all initial states $x(t)$, modes $\sigma(t)$, and times $t \in \mathbb{N}$ in the initial condition set* (3)*.*

Proof. The optimal control problem (6) is feasible for an initial state $x(t) = x_{0|t}$ if there exists an input sequence $u_{k|t} \in \mathcal{U}_{\sigma_{0|t}}$ such that the state sequence $x_{k|t} \in \mathcal{X}_{\sigma_{0|t}}$ produced by the dynamics (1a) in mode $\sigma(t_0) = \sigma_{0|t} \in \mathbb{I}$ is feasible and $x_{\delta|t} \in \mathcal{C}^\infty_{\sigma_{0|t}}$ i.e. $x_{0|t} \in \mathrm{Pre}^{\delta_{0|t}}_{\sigma_{0|t}}\big(\mathcal{C}_{\sigma_{0|t}}\big)$. □

According to Corollary 2, if the terminal sets $\mathcal{T}_{\sigma_{0|t}} = \mathcal{C}^\infty_{\sigma_{0|t}}$ are the maximal switch-RCI sets then MPC (7) guarantees constraint satisfaction for every initial condition for which it is possible to satisfy constraints. Thus, in terms of constraint satisfaction, the MPC (7) has no conservativeness.

4.2 Short-Horizon MPC

The requirement that the prediction horizon $N \geq d_i$ is longer than the dwell-times d_i can often make the optimal control problem (6) computationally intractable. In this section, we present an alternative to reduce the horizon of (6) at the expense of additional memory.

The predecessor-sets $\mathcal{P}^k_i = \mathrm{Pre}^k_i(\mathcal{C}_i)$ for $k = 0, \dots, d_i - N$ are computed offline and stored. The following terminal constraint is added to the optimal control problem (6)

$$x_{N|t} \in \mathcal{P}^s_{\sigma_{0|t}}. \tag{8}$$

where $s = \max\{\delta_{0|t} - N, 0\}$. The following corollary shows that adding the constraint (8) to the optimal control problem (6) ensures that the domain does not depend on the prediction horizon N.

Corollary 3. *The domain of problem* (6) *with constraint* (8) *is the same of the domain of the problem* (6) *with horizon $N \geq d_i$.*

Proof. If $\delta_{0|t} \leq N$ then the constraint (8) is redundant since the constraint (6d) covers $x_{N|t} \in \mathrm{Pre}^0_i(\mathcal{C}_i) = \mathcal{C}_i$ where $s = 0$. Thus, problem (6) with constraint (8) is feasible if and only if $x_{0|t} \in \mathrm{Pre}^{\delta_{0|t}}_{\sigma_{0|t}}(\mathcal{C}_{\sigma_{0|t}})$.

If $\delta_{0|t} > N$ then $s = \delta_{0|t} - N$. Thus, problem (6) with constraint (8) is feasible if and only if there exists $u_k \in \mathcal{U}_{\sigma_{0|t}}$ such that $x_{k+1|t} = f_{\sigma_{0|t}}(x_{k|t}, u_{k|t}) \in \mathcal{X}_{\sigma_{0|t}}$ and $x_{N|t} \in \mathcal{P}^s_{\sigma_{0|t}} = \mathrm{Pre}^{\delta_{0|t}-N}_{\sigma_{0|t}}(\mathcal{C}_{\sigma_{0|t}})$ for $k = 0, \dots, N$. From the recursive definition (2) of $\mathrm{Pre}(\cdot)$, problem (6) with constraint (8) is feasible if and only if $x_{0|t} \in \mathrm{Pre}^N_{\sigma_{0|t}}(\mathcal{P}^s_{\sigma_{0|t}}) = \mathrm{Pre}^{\delta_{0|t}}_{\sigma_{0|t}}(\mathcal{C}_{\sigma_{0|t}})$. □

According to Corollaries 2 and 3 we can design an MPC with the largest possible domain regardless of the prediction horizon N. However, this approach requires additional memory to store the predecessor-sets $\mathcal{P}^k_i$. Moreover, shortening the horizon can degrade the closed-loop performance of the controller.

5 Case Study: Vehicle lane-changing

In this section, we demonstrate the proposed approach on a case study of vehicle lane-changing control. The lateral vehicle dynamics are modeled in continuous-time by

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 1 \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{A_i} \begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{bmatrix}}_{B_i} u \tag{9}$$

where the state $x = [y, \dot{y}, \psi, \dot{\psi}]^\intercal$ includes the lateral position y, lateral velocity $\dot{y}$, yaw angle ψ, and yaw rate $\dot{\psi}$, and the steering angle u is the control input. The longitudinal velocity, and therefore the elements of A_i and B_i are mode-dependent [39].

The vehicle has six modes $\mathbb{I} = \{1, 2, 3, 4, 5, 6\}$. Modes $1, 2 \in \mathbb{I}$ are lane-keeping modes for the right and left lanes respectively. Modes $3, 4 \in \mathbb{I}$ are transition modes for left-to-right lane changing and the opposite, respectively. In modes $i = 1, \ldots, 4$ the longitudinal velocity is $v_{xi} = 80$ kilometers-per-hour (KPH). In mode $i = 5$ the vehicle keeps the right lane with a lower speed $v_{x5} = 60$KPH. While in mode $i = 6$ the vehicle keeps the left lane at higher velocity $v_{x6} = 100$KPH. In each mode $i \in \mathbb{I}$, the vehicle dynamics (9) are converted to discrete-time with sampling period $T_s = 0.2$ seconds (S).

The constraints for mode $1 \in \mathbb{I}$ enforce driving inside the right lane $\mathcal{X}_1 = \{x : 1 \leq y \leq 2\}$, as the constraints for mode $2 \in \mathbb{I}$ for the left lane $\mathcal{X}_2 = -\mathcal{X}_1$. The constraints for modes $3, 4 \in \mathbb{I}$ enable driving in both lanes $\mathcal{X}_3 = \mathcal{X}_4 = \{x : -2 \leq y \leq 2\}$, thus allowing the vehicle to move between the disjoint sets $\mathcal{X}_1 \cap \mathcal{X}_2 = \varnothing$, which is necessary for constraint satisfaction, according to Lemma 2. Modes $5, 6 \in \mathbb{I}$ are subject to the same constraints as modes $1, 2 \in \mathbb{I}$, respectively, $\mathcal{X}_5 = \mathcal{X}_1$ and $\mathcal{X}_6 = \mathcal{X}_2$. The input constraints $\mathcal{U}_i = \{u : |u| \leq 0.1\}$ are identical for all modes $i \in \mathbb{I}$. Each constraint set $\{\mathcal{X}_i\}_{i=\mathbb{I}}$ contains equilibria of the dynamics (9) of the form $x = [y, 0, 0, 0]^\intercal$ with equilibrium input $u = 0$.

The admissible mode switches are given by the graph $\mathbb{G} = (\mathbb{I}, \mathbb{E})$ shown in Fig. 1. The vehicle cannot directly switch between lanes, but rather it must switch from the right-lane keeping mode $i = 1$ to the right-to-left transition mode $i = 4$ and finally to the left-lane mode $i = 2$. Thus $(1, 4), (4, 2) \in \mathbb{E}$ and similarly $(2, 3), (3, 1) \in \mathbb{E}$. In addition, the vehicle can speed-up and slow-down in the right lane $(1, 5), (5, 1) \in \mathbb{E}$ and left lane $(2, 6), (6, 2) \in \mathbb{E}$. The transition modes $3, 4$ have a minimum dwell-time of 2.4S to allow lane changing, while the other modes have no restriction.

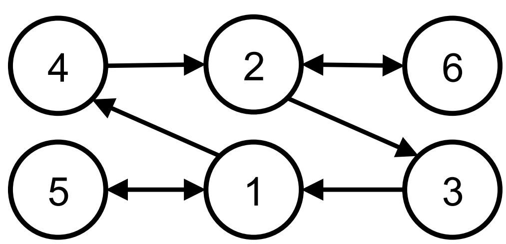

Figure 1: Graph $\mathbb{G} = (\mathbb{I}, \mathbb{E})$ of admissible switches between modes. Edges $(i, j) \in \mathbb{E}$ represent when mode switches $i \to j$ is allowed.

Fig. 2 shows slices of the control invariant sets $\mathcal{C}_i$ and initial condition sets $\mathrm{Pre}_i^{d_i}(\mathcal{C}_i)$ for each mode $i \in \mathbb{I} = \{1, 2, 3, 4, 5, 6\}$, obtained for zero yaw angle and yaw rate, $\psi = \dot{\psi} = 0$.

The switch-RCI sets for the right lane-keeping mode $1 \in \mathbb{I}$ and the left-to-right lane-transition mode $3 \in \mathbb{I}$ are the same $\mathcal{C}_1 = \mathcal{C}_3$. However, the reachable sets are different $\mathrm{Pre}_i^{d_1}(\mathcal{C}_3) \supset \mathrm{Pre}_1^{d_1}(\mathcal{C}_1)$ since the constraints are relaxed $\mathcal{X}_3 \supset \mathcal{X}_1$ for the transition mode $3 \in \mathbb{I}$ to allow the vehicle to move from the left to the right lane. The reachable set $\mathrm{Pre}_1(\mathcal{C}_1)$ for the right-lane mode $1 \in \mathbb{I}$ is the control invariant set $\mathcal{C}_1 = \mathrm{Pre}_1(\mathcal{C}_1)$, i.e., the only states that will not violate the lane-keeping are those inside $\mathcal{C}_1$.

Since $\mathcal{C}_1 = \mathrm{Pre}_1(\mathcal{C}_1)$, Definition 1 implies $\mathcal{C}_5 \subset \mathcal{C}_1 \subset \mathrm{Pre}(\mathcal{C}_5)$ where Figure 2 shows that the set inclusions are strict. In fact, the allowed states in the the slow-mode $5 \in \mathbb{I}$ are restricted $\mathcal{C}_5 \subset \mathcal{C}_1$ because the longitudinal

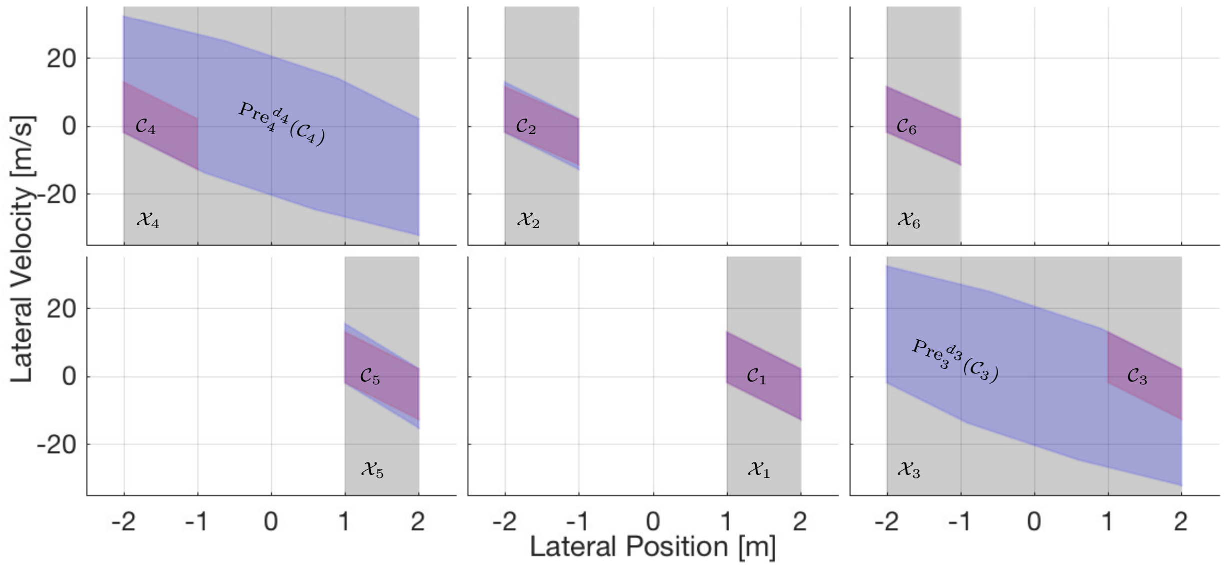

Figure 2: Sets for each mode $i \in \mathbb{I} = \{1,2,3,4,5,6\}$. Grey sets are state constraint sets $\mathcal{X}_i$ i.e. the lane boundaries, redsets are control invariant sets $\mathcal{C}_i$, and blue sets are initial condition sets $\mathcal{X}_i^0 = \mathrm{Pre}^d(\mathcal{C}_i)$. Subfigures for each mode arrange in same pattern as Figure 1.

velocity is allowed to suddenly increase, resulting in less stable dynamics that the controller must suddenly accommodate. Conversely, the possibility of switching into the slow-mode dynamics $5 \in \mathbb{I}$ does not restrict $\mathcal{C}_1 \subset \mathrm{Pre}_5(\mathcal{C}_5)$ the behavior of the right-lane keeping mode $1 \in \mathbb{I}$ since it becomes easier to satisfy the lane-keeping constraints due to the lower-speed. A similar relationship holds for $\mathcal{C}_2 \subset \mathcal{C}_6 \subset \mathrm{Pre}_2(\mathcal{C}_2)$.

The switch-RCI sets $\{\mathcal{C}_i\}_{i=1,2,3,4,5,6}$ are exploited to design an MPC for lane-changing. The MPC computes the control input by solving (6) where the mode-dependent terminal constraint set (6d) is the control switch-invariant set $\mathcal{C}_{\sigma(t)}$ for the current mode $\sigma(t) \in \mathbb{I}$. The horizon of the MPC is $N = \max\left\{\frac{d_i}{T_s}\right\} = 12$. The terminal and stage costs are mode-dependent and given by

$$\begin{aligned} p_i(x) &= \|x - r_i\|_{P_i}^2 \\ q_i(x,u) &= \|x - r_i\|_Q^2 + \|u\|_R^2, \end{aligned}$$

where the mode-dependent references $r_1 = r_3 = r_5 = 1.5$ and $r_2 = r_4 = r_6 = -1.5$ are the centerlines of the right and left lanes respectively. The penalty matrices Q and R were chosen to provide reference tracking and a smooth transition between lanes. The terminal cost matrices P_i are the infinite-horizon cost-to-go matrix for the corresponding linear quadratic regulator (LQR) with the i-th modes dynamics i.e. (A_i, B_i).

Fig. 3 shows the lateral position of the vehicle in closed-loop with a switched LQR and switched MPC controllers where the mode was selected by a high-level path-planner, in this case an invariant-set path-planner [36–38]. While the LQR provides smooth tracking of the reference, it does not satisfy the constraints specified by the path-planner, which may result in a collision. Similarly, if the switch-invariant terminal constraints (6d) are omitted, the MPC problem (6) could become infeasible after a mode switch. Instead, the MPC (6) with the terminal constraint (6d), guarantees constraint satisfaction according to Theorem 3. This is confirmed by Fig. 3, which shows that the vehicle satisfies the lane-keeping constraints specified by the path-planner. Thus, the theoretical contributions of this paper enabled the design of an MPC that retains the smooth reference tracking of LQR while ensuring that the lane-keeping constraints specified by the path-planner are always satisfied.

6 Conclusions

This paper derived necessary and sufficient conditions for guaranteeing constraint satisfaction in constrained switched systems where the dwell-time and admissible mode switches are restricted. The conditions were

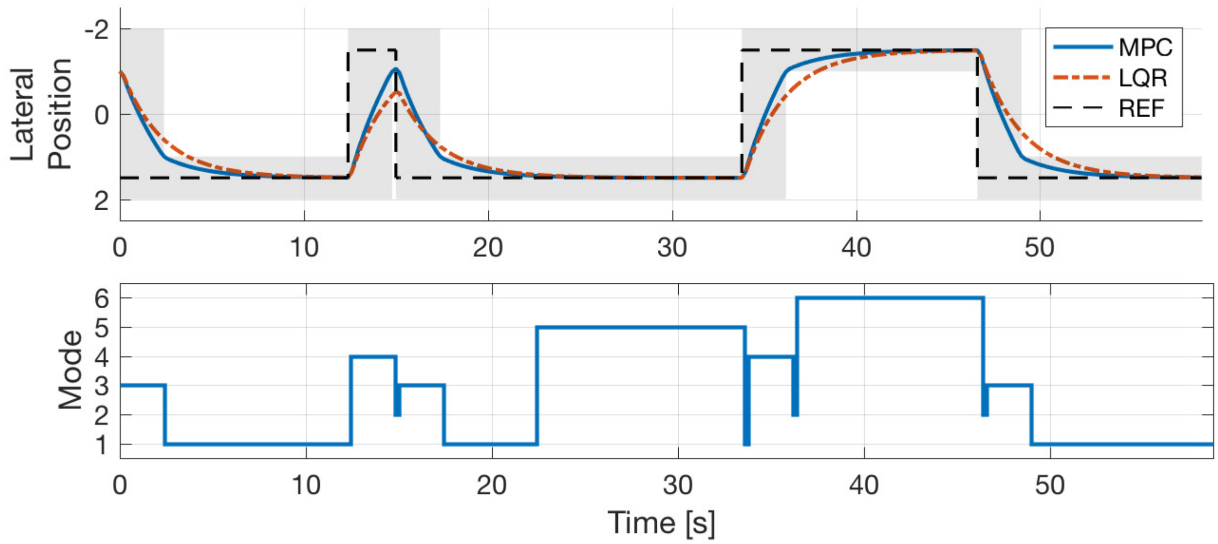

Figure 3: Lateral position $y(t)$ and mode $\sigma(t)$ for a vehicle during a lane change maneuver. The shaded regions show the lane constraints at each time. Observe that the MPC achieves constraint satisfaction for all maneuvers, while the LQR does not.

derived using switch-RCI sets that are constraint admissible control invariant set with the additional property that constraints are enforce during the transit after a mode switch. The switch-RCI sets were used to design persistently feasible model predictive controllers. The switch-RCI sets and MPC were demonstrated on a vehicle lane-changing case study in which the dynamics and constraints of the vehicle were selected by a high-level path-planner. The only restriction that the controller imposed on the path-planner to ensure constraint enforcement, was the minimum amount of time between lane change requests.

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