Robustifying the Kalman Filter against Measurement Outliers: An Innovation Saturation Mechanism

Fang, H.; Haile, M.; Wang, Y.

TR2018-173 December 29, 2018

Abstract

Measurements made on a practical system can often be subject to outliers due to sensor errors, changes in ambient environment, data loss or malicious cyber attacks. The outliers can seriously reduce the accuracy of the Kalman filter (KF) when it is applied for state estimation. This paper proposes an innovation saturation mechanism to robustify the standard KF against outliers. The basic notion is to saturate an innovation when it is distorted by an outlier, thus preventing it from impairing the state estimation process. The mechanism presents an adaptive adjustment of the saturation bound. The design is performed for both continuous- and discretetime systems, provably leading to bounded-error estimation given bounded outliers. Numerical simulation further shows the efficacy of the proposed design. Compared to many existing methods, the proposed design is computationally efficient and amenable to practical implementation, and also requires neither measurement redundancy nor extensive manual tuning.

IEEE Conference on Decision and Control (CDC)
Robustifying the Kalman Filter against Measurement Outliers: An Innovation Saturation Mechanism

Huazhen Fang, Mulugeta A. Haile and Yebin Wang

Abstract—Measurements made on a practical system can often be subject to outliers due to sensor errors, changes in ambient environment, data loss or malicious cyber attacks. The outliers can seriously reduce the accuracy of the Kalman filter (KF) when it is applied for state estimation. This paper proposes an innovation saturation mechanism to robustify the standard KF against outliers. The basic notion is to saturate an innovation when it is distorted by an outlier, thus preventing it from impairing the state estimation process. The mechanism presents an adaptive adjustment of the saturation bound. The design is performed for both continuous- and discrete-time systems, provably leading to bounded-error estimation given bounded outliers. Numerical simulation further shows the efficacy of the proposed design. Compared to many existing methods, the proposed design is computation efficiently and amenable to practical implementation, and also requires neither measurement redundancy nor extensive manual tuning.

I. INTRODUCTION

Arguably the most celebrated estimation approach, the Kalman filter (KF) has found use in numerous applications across the fields of control systems, signal processing, system health monitoring, navigation and econometrics [1]. Its popularity is partially attributed to the optimality—for a linear system, it is optimal among all filters when certain Gaussian noise assumptions are satisfied [2]. However, the performance of the KF can be seriously degraded by measurement outliers. In practice, outliers can come from a diversity of sources, e.g., unreliable sensors, environmental variability, data dropouts in transmission, and data falsification attacks from cyberspace [3–6]. They can easily cause the KF to deviate from a normal course, at least temporarily, and may even trigger a complete failure in extreme cases.

How to robustify the KF against measurement outliers has attracted significant interest from researchers during the past years. Existing methods can be mainly divided into three categories. The first category models the measurement noises using heavy-tailed distributions rather than commonly used exponential distributions, e.g., the Gaussian distribution as used in the classical KF, to capture the occurrence of an outlier. For instance, heavy-tailed Gaussian-mixture [7; 8] and $t$-distributed noise models [9] are used to modify the KF for better robustness. In [10], an outlier is viewed as a result of measurement noises with variable covariances.

Assuming the noise covariances to follow an inverse Gamma distribution, it proposes to add to the KF a procedure of adaptive identification of the key parameters involved in the inverse Gamma distribution. Methods of the second category seek to assign the measurement sample at each time instant with a weight, in an attempt of downweighting outlying measurements. In [3], an expectation-maximization algorithm is used to enable adaptive determination of the weight for a measurement. The results are generalized in [11] and extended to the smoothing problem. In [12; 13], a measurement-weighting-based prewhitening procedure is designed to decorrelate outliers from normal measurements as a basis for building a robust KF technique. The third category considers simultaneous state and outlier estimation, which regards an outlier as an input to the system’s measurement process. The literature includes a few studies based on minimum variance unbiased estimation [14–17]. In addition, Bayesian methods are developed in [18; 19] to achieve joint state and outlier estimation from a probabilistic perspective. Despite the usefulness, these robust KF techniques come with a dramatic increase in computational complexity, due to the needed iterative optimization or other computationally expensive procedures. Besides, they often require redundant measurements in order to either differentiate outliers from normal measurements or estimate them directly. This, however, is not always possible, because a real system often allows only a limited number of sensors to be deployed.

In addition to the above outlier-robust KF methods, one can also find some other types of estimation approaches in the literature capable of suppressing outliers. Among them, a well-known one is $H_{\infty}$ filtering [20; 21], which considers outliers as unknown yet bounded uncertainty. However, this approach may introduce significant conservatism as it performs worst-case estimation by design. A stubborn observer is developed in [22], which employs a saturation function in the output injection signal to mitigate the influence of outliers. This method is not only computationally fast but also can deal with large outliers.

In this work, we propose to robustify the classical KF with an innovation saturation mechanism. The innovation plays a key role in correcting the state prediction in the KF but can be distorted by outliers. To overcome such a vulnerability, our mechanism saturates the innovation when it is unreasonably large in order to suppress the effects of outliers. The design of the mechanism includes a procedure for adaptively adjusting the saturation bound to effectively grasp the change of the innovation. Along this line, we develop the saturated KF (SKF) approaches for both continuous- and discrete-time
systems. It can be proven that the SKF would produce bounded-error estimation when the outliers are bounded in magnitude. Numerical simulation further illustrates the effectiveness. This work is a generalization of [22] to the KF. Compared to methods in the literature, the proposed SKF approaches are structurally concise, computationally efficient, and robust against even very large outliers, thus lending itself well to practical application.

This paper is organized as follows. Section II develops the SKF for linear continuous-time systems and analyzes its stability. Section III extends the results to discrete-time systems. A numerical example is provided in Section IV to illustrate the usefulness of the proposed design. Finally, our concluding remarks are presented in Section V.

Notation: Notations used throughout the paper are standard. The $n$-dimensional Euclidean space is denoted as $\mathbb{R}^n$. For a vector, $\| \cdot \|$ denotes its 2-norm. The notation $I$ is an identity matrix; $X > 0$ ($\geq 0$) means that $X$ is a real, symmetric and positive definite (semidefinite) matrix; for a symmetric block matrix, we use a star ($\ast$) to represent a symmetry-induced block; the notation diag$((\ldots))$ stands for a block-diagonal matrix. The minimum and maximum eigenvalues of a real, symmetric matrix are denoted as $\lambda(\cdot)$ and $\bar{\lambda}(\cdot)$. Matrices are assumed to be compatible for algebraic operations if their dimensions are not explicitly stated.

II. SKF FOR CONTINUOUS-TIME SYSTEMS

This section develops the SKF approach for a linear continuous-time system and then analyzes its stability.

A. SKF Design

Consider the following model

$$
\begin{align*}
\dot{x}_t &= Ax_t + w_t, \\
y_t &= Cx_t + Dd_t + v_t,
\end{align*}
$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ the measurement vector, and $w_t, v_t \in \mathbb{R}^n$ and $d_t \in \mathbb{R}^m$ zero-mean, mutually independent noises with covariances given by $Q \geq 0$ and $R > 0$, respectively. Note that the measurement $y_t$ is subjected to the outlier effects caused by a disturbance $d_t \in \mathbb{R}^m$. As often needed for estimation, it is assumed that $(A, C)$ is detectable and $(A, Q^{1/2})$ stabilizable.

Modifying the classical KF, we propose the following state estimation procedure:

$$
\begin{align*}
\dot{x}_t &= A\hat{x}_t + K_t \cdot \text{sat}_\sigma(y_t - C\hat{x}_t), \\
K_t &= P_t C^T R^{-1}, \\
\dot{P}_t &= AP_t + P_t A^T + Q - K_t R K_t^T,
\end{align*}
$$

where $K_t$ is the estimation gain matrix, and $P_t$ a positive definite matrix that represents the estimation error covariance. Note that, for the classical KF, the state estimation is corrected by the innovation $(y_t - C\hat{x}_t)$. Its effectiveness, however, can be compromised if $y_t$ is corrupted by an outlier. To overcome this issue, we draw analogy to the design of a stubborn observer in [22] and use a saturated innovation instead, as shown in (2a). Specifically, it is defined as

$$
\text{sat}_\sigma(y_t - C\hat{x}_t) = \begin{bmatrix} \text{sat}_{\sigma_1}(y_{1,t} - C_{11}\hat{x}_1) \\ \vdots \\ \text{sat}_{\sigma_p}(y_{p,t} - C_{p1}\hat{x}_1) \end{bmatrix},
$$

where $\sigma_1 > 0$, $y_{i,t}$ is the $i$-th element of $y$, and $C_{i1}$ the $i$-th row of $C$. For a variable $r$, the saturation function is defined as $\text{sat}_\sigma(r) = \max\{ -\epsilon, \min(\epsilon, r) \}$. For (3), one can approximately view the saturation range $[-\sqrt{\sigma_i}, \sqrt{\sigma_i}]$ as an anticipated range of the innovation. If falling within this range, the innovation is considered reasonable and applied without change to update the state estimation. Otherwise, it may be affected by an outlier and thus saturated so as not to mislead the estimation.

It is observed that a fixed saturation bound can be limited in efficacy of rejecting outliers as it may confuse with an outlier a certain measurement generating a large innovation, or vice versa. In addition, it can also be difficult in practice to select a fixed bound, especially when knowledge about possible outliers afflicting a system is scarce. The following procedure is thus introduced to adaptively adjust the saturation bound:

$$
\dot{\text{sat}}_{\sigma_i}(y_{i,t} - C_i\hat{x}_t) = \mu_i \sigma_i (y_{i,t} - C_i\hat{x}_t)^2, \quad \sigma_{i,0} > 0,
$$

for $i = 1, 2, \ldots, p$, where $\mu_i < 0$ and $\gamma_i > 0$. We define $\Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_p)$. With (4), $\sigma_i$ can dynamically change according to the innovation $(y_{i,t} - C_i\hat{x}_t)$. This will enable an improved discernment between an outlier and a normal measurement.

Putting together (2)-(4), we obtain the SKF for the linear continuous-time system in (1). Next, we analyze the its stability properties.

B. Stability Analysis

Define the state estimation error as $e_t = \hat{x}_t - x_t$. The dynamics of $e_t$ is governed by

$$
\dot{e}_t = A e_t - K_t \cdot \text{sat}_\sigma(C e_t - D d_t - v_t) - w_t.
$$

To proceed further, we define the following matrix

$$
S_t = \begin{bmatrix}
M_t - \alpha P_t^{-1} & -C^T (R_t^{-1} + W) & C^T (\Gamma_t^{-1} - R_t^{-1}) D \\
* & 2W & WD \\
* & * & U
\end{bmatrix},
$$

where $M_t = P_t^{-1} Q P_t^{-1} + C^T (R_t^{-1} - \Gamma_t) C$, $W$ is a diagonal positive definite matrix, $U$ a positive definite matrix, and $\alpha > 0$ a positive scalar. Furthermore, we recall a well-known fact [23]: if $(A, Q^{1/2})$ is stabilizable and $(A, C)$ detectable, then $P_t$ for $P_0 \geq 0$ in (2c) will approach a unique positive-definite solution $\tilde{P}$ satisfying

$$
AP_\infty + P_\infty A^T = Q - P_\infty C^T R^{-1} C P_\infty = 0.
$$

We also assume $w_t = 0$ and $v_t = 0$ to perform the stability analysis in a deterministic setting. Hence, $Q$ and $R$ would be considered as weight matrices rather than covariances.
The following result is obtained about the stability of the proposed SKF:

Theorem 1: Suppose \( w_i = 0, v_i = 0 \) and \( \|d_i\| \leq \delta < \infty \), where \( \delta > 0 \). If there exist \( P_o, W, U, \alpha \) and \( \Gamma \) such that \( S(t) \geq 0 \) and \( 0 < \alpha \leq -\max(\mu_i) \) for \( i = 1, 2, \ldots, p \), then the estimation error \( e_t \) is upper-bounded with

\[
\|e_t\| \leq \sqrt{\frac{1}{c_2} \left[ e^{-\alpha t} V_0 + \frac{1}{\alpha} (1 - e^{-\alpha t}) c_1 \delta^2 \right]}, \tag{6a}
\]

\[
\lim_{t \to \infty} \|e_t\| \leq \sqrt{\frac{1}{c_2} \alpha c_3 \delta}, \tag{6b}
\]

where \( c_1 = \frac{\lambda(\bar{U} + D^T \Gamma D)}{\alpha} \), \( c_2 = \frac{\lambda(P^{-1}_{1})}{} \) and \( c_3 = \frac{\lambda(P^{-1}_{\infty})}{} \).

Proof: We consider using the Lyapunov function

\[ V_t = e_t^T P_t^{-1} e_t + \sum_i \sigma_i x_{i,t}. \]

The first-order time derivative of \( V_t \) along (5) is

\[
\dot{V}_t = 2 e_t^T P_t^{-1} \dot{e}_t + e_t^T \frac{d}{dt} \left( P_t^{-1} \right) e_t + \sum_i \dot{\sigma}_i x_{i,t}
= 2 e_t^T P_t^{-1} [A e_t - K_i \cdot \sigma_i (C e_t - D d_t)]
+ e_t^T P_t^{-1} \left( A P_t + P_t A^T + Q - K_i R K_i^T \right) P_t^{-1} e_t
+ \sum_i \mu_i \sigma_i \cdot \sigma_i + (C e_t - D d_t)^T \Gamma (C e_t - D d_t)

= -e_t^T P_t^{-1} Q P_t^{-1} e_t + e_t^T C^T (R^{-1} + \Gamma) C e_t
- 2 e_t^T C^T R^{-1} \cdot \sigma_i (C e_t - D d_t) - e_t^T C^T \Gamma D d_t
+ d_t^T D^T \Gamma D d_t + \sum_i \mu_i \sigma_i \cdot \sigma_i.
\]

Let us define \( s_t = C e_t - D d_t - \sigma_i (C e_t - D d_t) \). Then,

\[
\dot{V}_t = -e_t^T M_e e_t + 2 e_t^T C^T \cdot \sigma_i (C e_t - D d_t) + e_t^T C^T (R^{-1} - \Gamma) D d_t
+ d_t^T D^T \Gamma D d_t + \sum_i \mu_i \sigma_i \cdot \sigma_i.
\]

By [24, Lemma 1.6], we have

\[ s_t^t W (s_t - C e_t + D d_t) \geq 0. \]

It then follows that

\[
\dot{V}_t \leq \dot{V}_t - 2 s_t^t W(s_t - C e_t + D d_t)
= -e_t^T M_e e_t + 2 e_t^T C^T \cdot (R^{-1} + \Gamma) D d_t - 2 s_t^t W s_t
+ 2 e_t^T C^T (R^{-1} - \Gamma) D d_t + s_t^t W D d_t + \sum_i \mu_i \sigma_i \cdot \sigma_i.
\]

\[
\leq -e_t^T M_e e_t + 2 e_t^T C^T (R^{-1} + \Gamma) D d_t
+ \sum_i \mu_i \sigma_i \cdot \sigma_i.
\]

If \( S(t) \geq 0 \), we have

\[
\dot{V}(t) \leq -\alpha e_t^T P_t^{-1} e_t \alpha \sum_i \sigma_i \cdot \sigma_i + \sum_i (\mu_i + \alpha) \sigma_i \cdot \sigma_i
+ d_t^T (U + D^T \Gamma D) d_t.
\]

If \( 0 < \alpha \leq -\max(\mu_i) \), one has \( \mu_i + \alpha \leq 0 \). Then,

\[
\dot{V}_t \leq -\alpha V_t + d_t^T (U + D^T \Gamma D) d_t
\leq -\alpha V_t + c_1 d_t^2.
\]

Hence,

\[
V_t \leq e^{-\alpha t} V_0 + \frac{1}{\alpha} (1 - e^{-\alpha t}) c_1 \delta^2.
\]

Furthermore, \( V_t \geq c_2 \|e_t\|^2 \). Then,

\[
\|e_t\|^2 \leq \frac{1}{c_2} \left[ e^{-\alpha t} V_0 + \frac{1}{\alpha} (1 - e^{-\alpha t}) c_1 \delta^2 \right],
\]

which implies (6a). When \( t \to \infty \), we can obtain (6b).

Theorem 1 shows that, if the system is noise-free and the outliers are upper-bounded, the proposed SKF can lead to bounded-error estimation under certain conditions. Meanwhile, it is easy to verify that \( e_t \) will exponentially approach zero as \( t \to \infty \) if \( d_t = 0 \).

Remark 1: For the proposed SKF, selection of \( P_0 \) is critical to make the condition \( S_t \geq 0 \) satisfied. Given the structure of \( S_t \), it is cautioned that too large a \( P_0 \) may bring the risk of divergent estimation. However, note that \( P_t \) is monotonically non-decreasing for \( P_0 = 0 \) if \( (A, Q^2) \) is stabilizable and \( (A, C) \) detectable [23]. Leveraging this property, we recommend that \( P_0 \) be set to zero or close to zero when the SKF approach is to be implemented.

III. SKF FOR DISCRETE-TIME SYSTEMS

Extending the notion in Section II, this section investigates the development of SKF for linear discrete-time systems.

A. SKF Design

Consider a discrete-time system

\[
\begin{align*}
\dot{x}_{k+1} &= A x_k + w_k, \\
y_k &= C x_k + D d_k + v_k.
\end{align*}
\]

(7)

The notations in above are the same as in Section II. Still, the noises \( w_k \) and \( v_k \) are zero-mean, mutually independent with covariances \( Q \geq 0 \) and \( R > 0 \), respectively. We also assume that \( (A, Q^2) \) is stabilizable and \( (A, C) \) detectable without loss of generality. In addition, we assume that \( A \) is invertible.

For this system, we propose the SKF as follows:

\[
\dot{x}_{k|k-1} = A \hat{x}_{k-1|k-1},
\]

(8a)

\[
P_{k|k-1} = AP_{k-1|k-1} A^T + Q,
\]

(8b)

\[
\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \cdot \sigma_i (y_k - C \hat{x}_{k|k-1}),
\]

(8c)

\[
K_k = P_{k|k-1} C^T (CP_{k|k-1} C^T + R)^{-1},
\]

(8d)

\[
P_{k|k} = P_{k|k-1} - K_k (CP_{k|k-1} C^T + R) K_k^T,
\]

(8e)

where \( \hat{x}_{k|k} \) is the one-step-forward prediction of \( x_k \), and \( \hat{x}_{k|k} \) is the updated estimate when \( y_k \) arrives to correct the prediction. In addition, \( K_k \) is the estimation gain, and \( P_{k|k} \) and \( P_{k|k-1} \) the estimation error covariances in the standardKF. Akin to (2), an innovation saturation mechanism is used to deal with possible measurement outliers, as shown in (8c). The saturation bound is dynamically adjusted by

\[
\sigma_{i,k+1} = \mu_i \sigma_{i,k} + \gamma_i (y_{i,k} - C_i \hat{x}_{k|k-1})^2, \quad \sigma_{i,0} > 0,
\]

(9)

for \( i = 1, 2, \ldots, p \) where \(-1 < \mu_i < 1 \) and \( \gamma_i > 0 \).
B. Stability Analysis

Now, let us consider the stability analysis for the above SKF. Defining the state prediction error as \( e_k = \hat{x}_{k|k-1} - x_k \), its dynamics can be expressed as

\[
e_{k+1} = A_0 + A K_k \cdot \text{sat}_\sigma (C e_k - D d_k - v_k) - w_k.
\]

Before proceeding further, we show some results that will be needed later. Given that \((A, Q^T)\) is stabilizable and that \((A, C)\) detectable, \( P_{k|k-1} \) will converge to a fixed positive definite matrix \( P_\infty \) satisfying

\[
P_\infty = A P_\infty A^T + Q - A P_\infty C^T (C P_\infty C^T + R)^{-1} \cdot C P_\infty A^T.
\]

It is also known that \( P_{k|k-1} \) is upper and lower bounded, and so is \( P_{k|k} \). Hence, there should exist an \( \epsilon > 0 \) such that \( P_{k|k}^{-1} \leq \epsilon I \). Then,

\[
P_{k|k|k}^{-1} = (A P_{k|k} A^T + Q)^{-1} = A^{-1} \cdot P_{k|k}^{-1} \cdot A^{-1}
\]

\[
\leq A^{-1} \cdot P_{k|k}^{-1} (A^T Q A^{-1})^{-1} \cdot P_{k|k}^{-1} A^{-1}
\]

\[
\leq A^{-1} \cdot P_{k|k}^{-1} (\epsilon I + A^T Q A^{-1})^{-1} \cdot P_{k|k}^{-1} A^{-1}
\]

where \( Q = (\epsilon I + A^T Q A^{-1})^{-1} \).

We also define

\[
Z_k = \begin{bmatrix}
T_{1,k} & -\alpha P_{k|k}^{-1} & T_{2,k} - C^T W & T_{3,k} \\
* & T_{4,k} + 2W & T_{5,k} + WD & U
\end{bmatrix},
\]

where \( W \) is a diagonal positive definite matrix, \( U \) a positive definite matrix, \( \alpha > 0 \) a positive scalar, and

\[
T_{1,k} = C^T R^{-1} C + P_{k|k}^{-1} Q P_{k|k} - P_{k|k}^{-1} Q C^T R^{-1} C
\]

\[
- C^T R^{-1} C Q P_{k|k} - C^T R^{-1} C (P_{k|k} - Q) \cdot C^{-1} - C^T R^{-1} C - C^T \Gamma C
\]

\[
T_{2,k} = -C^T R^{-1} + P_{k|k}^{-1} Q C^T R^{-1}
\]

\[
+ C^T R^{-1} C (P_{k|k} - Q) C^T R^{-1},
\]

\[
T_{3,k} = \left[-C^T R^{-1} + P_{k|k}^{-1} Q C^T R^{-1}
\right.\]

\[
+ C^T R^{-1} C (P_{k|k} - Q) C^T R^{-1} + C^T \Gamma D,
\]

\[
T_{4,k} = -R^{-1} C (P_{k|k} - Q) C^T R^{-1},
\]

\[
T_{5,k} = -[R^{-1} C (P_{k|k} - Q) C^T R^{-1} + C^T \Gamma D].
\]

In addition, we consider another matrix

\[
T_{0,k} = D^T \left[ R^{-1} C (P_{k|k} - Q) C^T R^{-1} + \Gamma \right] D.
\]

The following theorem shows the result about the stability of the prediction error dynamics.

**Theorem 2:** Suppose \( w_k = 0 \), \( v_k = 0 \), \( \|d_k\| \leq \delta < \infty \) and that \( A \) is invertible. If there exist \( P_{0|1}, W, U, \alpha \) and \( \Gamma \) such that

\[
Z_k \geq 0 \text{ and } 0 < \alpha \leq 1 \cdot \max(\mu_i) \text{ for } i = 1, 2, \ldots, n,
\]

then the estimation error \( e_k \) is upper bounded with

\[
\|e_k\| \leq \sqrt{e_2 \left( 1 - \alpha \right) \epsilon V_0 + \frac{1 - (1 - \alpha)^{-1}}{\alpha} c_1 \delta^2},
\]

\[
l_k \rightarrow \infty \lim \|e_k\| \leq \sqrt{\epsilon_0 c_2 c_3 \delta},
\]

where \( \epsilon_0 = \lambda_0 (T_w + U) \), \( \epsilon_2 = \lambda_0 (P_{k|k-1}^{-1}) \), and \( c_3 = \Delta (P_{k|k-1}^{-1}) \).

**Proof:** Consider a Lyapunov function

\[
V_k = e_k^T P_{k|k-1}^{-1} e_k + \sum_i \sigma_i e_i.
\]

Using (10), we have

\[
V_{k+1} = e_k^T P_{k|k+1}^{-1} e_k + \sum_i \sigma_i e_i + \sum_i \sigma_i e_i
\]

\[
\leq \left[A e_k - A K_k \cdot \text{sat}_\sigma (C e_k - D d_k)\right]^T
\]

\[
\cdot A^{-1} \left[P_{k|k}^{-1} - P_{k|k}^{-1} Q P_{k|k}^{-1} \right] A^{-1} \cdot A e_k - A K_k \cdot \text{sat}_\sigma (C e_k - D d_k) + \sum_i \mu_i e_i,
\]

\[
= e_k^T \left[P_{k|k}^{-1} - P_{k|k}^{-1} Q P_{k|k}^{-1} \right] e_k + 2 e_k^T \left[P_{k|k}^{-1} - P_{k|k}^{-1} Q P_{k|k}^{-1} \right] K_k \cdot \text{sat}_\sigma (C e_k - D d_k)
\]

\[
+ (C e_k - D d_k) \cdot \text{sat}_\sigma (C e_k - D d_k) + \sum_i \mu_i e_i.
\]

Let us define

\[
s_k = C e_k - D d_k - \text{sat}_\sigma (C e_k - D d_k).
\]

In addition, we have

\[
P_{k|k}^{-1} = P_{k|k-1}^{-1} + C^T R^{-1} C, \quad P_{k|k}^{-1} K_k = C^T R^{-1} C \quad \text{and} \quad K_k P_{k|k}^{-1} K_k = R^{-1} C e_k C^T R^{-1} \cdot C e_k.
\]

These relations can be readily proven. It then follows that

\[
V_{k+1} \leq e_k^T P_{k|k-1}^{-1} e_k + 2 e_k^T T_{1,k} e_k + 2 e_k^T T_{2,k} s_k + 2 e_k^T T_{3,k} d_k
\]

\[
- s_k^T T_{4,k} s_k - s_k^T T_{5,k} d_k + d_k^T T_{6,k} d_k + \sum_i \mu_i e_i.
\]

According to (24, Lemma 1.6), we have

\[
-s_k^T W(s_k - C e_k + D d_k) \geq 0.
\]

It can be obtained that

\[
V_{k+1} \leq V_{k+1} - 2 s_k^T W(s_k - C e_k + D d_k)
\]

\[
= e_k^T P_{k|k-1}^{-1} e_k - e_k^T T_{1,k} e_k - 2 e_k^T T_{2} C^T W s_k
\]

\[
- 2 e_k^T T_{3,k} d_k + s_k^T (T_{4,k} + 2W) s_k
\]

\[
- 2 s_k^T (T_{5,k} + WD) d_k + d_k^T T_{6,k} d_k + \sum_i \mu_i e_i
\]

\[
= e_k^T P_{k|k-1}^{-1} e_k - s_k^T [\begin{bmatrix} T_{1,k} & T_{2,k} - C^T W & T_{3,k} \\
* & T_{4,k} + 2W & T_{5,k} + WD & U
\end{bmatrix} e_k]
\]

\[
+ d_k^T (T_{6,k} + U) d_k + \sum_i \mu_i e_i.
\]
If $Z_k \geq 0$, then
\[
V_{k+1} \leq (1 - \alpha)e_k^T P_{k|k-1}^{-1} e_k + (1 - \alpha) \sum_i \sigma_{i,k} \\
+ \sum_i (\mu_i + \alpha - 1) \sigma_{i,k} + d_k^T (T_{6,k} + U) d_k.
\]

Because $0 < \alpha \leq 1 - \max(\mu_i)$ for $i = 1, 2, \ldots, p$,
\[
V_{k+1} \leq (1 - \alpha)V_k + d_k^T (T_{6,k} + U) d_k \\
\leq (1 - \alpha)V_k + \bar{\lambda} (T_6 + U) \delta^2,
\]
from which one can easily obtain (11a)-(11b).

Remark 2: For the discrete-time case, it is also recommended that $P_{k|k-1}$ is initialized with a small $P_{0|0}$, which can be zero or near zero, in order to make the conditions in Theorem 2 satisfied more easily.

Remark 3: It is observed that stability analysis of the SKF approach for both continuous- and discrete-time systems involves simplifying assumptions, e.g., zero process and measurement noises and invertible $A$ (for the discrete-time case). The zero-noise assumption allows us to perform the analysis in a deterministic setting. This move makes the problem more tractable by avoiding the nonlinear, saturation-function-based transformation of random noise variables, which will appear otherwise in a stochastic setting and may obstruct a proof. Yet, this does not imply that the SKF approach can only be applied to deterministic systems. From a large number of simulations, we still consistently observe satisfactory estimation performance when noises exist. An example in this respect is also shown in Section IV. Besides, although the assumption that $A$ of a discrete-time system is invertible sounds strong, it is noticed that a zero-order-hold discretization of a linear continuous-time system will always result in an invertible $A$. How to relax these assumptions still presents interesting challenges, which will motivate our further work.

Remark 4: It is worth pointing out that different strategies can be developed and used to enable the saturation bound adjustment. For instance, one can modify (4) as
\[
\dot{x}_{i,t} = \mu_i \sigma_{i,t} + \gamma_i |y_{i,t} - C_i \hat{x}_i|,
\]
or (9) as
\[
\sigma_{i,k+1} = \mu_i \sigma_{i,k} + \gamma_i |y_{i,k} - C_i \hat{x}_{i,k-1}|,
\]
which should come with a different set of stability conditions. It will hence be interesting to investigate different methods for adjusting the saturation bound and assessing their performance.

IV. ILLUSTRATIVE EXAMPLE

In this section, we present a numerical simulation example to illustrate the effectiveness of the proposed SKF for discrete-time systems. Consider a discrete-time system as in (7) with
\[
A = \begin{bmatrix} 0.8 & 0.32 & 0 \\ 0 & -0.67 & 0.5 \\ 0.2 & -0.1 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 \\ -1.2 \\ 0.9 \end{bmatrix}, \\
C = \begin{bmatrix} 1.4 & 0.6 & 0.3 \end{bmatrix}, \quad D = 1.
\]

Suppose that the system is excited by a sinusoidal input $u_k = 2 \sin(k \pi T_s/2)$ with the sampling period $T_s = 0.01$ seconds. The noise covariances are given by $Q = 10^{-3} I$ and $R = 10^{-3}$, respectively. In addition, the measurement $y_k$ is subjected to occasional outliers, $d_k$, at the fourth, fifth, sixth, seventh and eighth seconds, with a magnitude of -7.25, 9, -1.5, 7.5 and -11.25. The profile of $y_k$ is depicted in Fig. 1, which shows that the outlier measurements considerably deviate from the norm.

Both the classical KF and the proposed SKF are applied to perform estimation for the considered system for a comparison. For the SKF, $\mu = 0.7$ and $\gamma = 1$. The initial $P_{0|0}$ is set as $P_{0|0} = 0.1 I$ for both filters. The estimation results are summarized in Fig. 2. It is observed from Fig. 2(a)-2(c) that the classical KF leads to satisfactory estimation for each state variable when there is no outlier. However, inaccurate and quite turbulent estimation results when an outlier arises, implying risks for practical system monitoring or control. For the SKF, it demonstrates estimation performance comparable to the classical KF when the system is running free of outliers. When outliers appear, it can still maintain a smooth and accurate estimation, thanks to the innovation saturation. This reflects a remarkable effectiveness of the SKF in suppressing the undesirable influence of outliers. The profile of $\sigma$ is illustrated in Fig. 2(d), which shows a dynamic change in response to the appearance of outliers.

V. CONCLUSION

The KF has gained wide application in numerous fields as one of the most popular state estimation tools. Its performance, however, can be reduced by measurement outliers due to sensor anomaly, data transmission errors or cyber attacks. Aiming to enhance robustness of the KF, this paper proposes to apply an innovation saturation mechanism to the standard KF. This mechanism saturates the innovation process, which is crucial for correcting the state estimation, to ensure that a reasonable correction is applied when outliers occur. The design is performed for both continuous- and discrete-time Kalman filtering. Theoretical analysis provides useful stability properties of the proposed approaches. Numerical
Fig. 2: Estimation of the state variables: (a) $x_1$ versus $\hat{x}_1$; (b) $x_2$ versus $\hat{x}_2$; (c) $x_3$ versus $\hat{x}_3$; change of $\sigma$.