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Fang, H.; Haile, M.; Wang, Y.

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Huazhen Fang, Mulugeta A. Haile and Yebin Wang

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I. INTRODUCTION

Arguably the most celebrated estimation approach, the Kalman filter (KF) has found use in numerous applications across the fields of control systems, signal processing, system health monitoring, navigation and econometrics [1]. Its popularity is partially attributed to the optimality—for a linear system, it is optimal among all filters when certain Gaussian noise assumptions are satisfied [2]. However, the performance of the KF can be seriously degraded by measurement outliers. In practice, outliers can come from a diversity of sources, e.g., unreliable sensors, environmental variability, data dropouts in transmission, and data falsification attacks from cyberspace [3–6]. They can easily cause the KF to deviate from a normal course, at least temporarily, and may even trigger a complete failure in extreme cases.

How to robustify the KF against measurement outliers has attracted significant interest from researchers during the past years. Existing methods can be mainly divided into three categories. The first category models the measurement noises using heavy-tailed distributions rather than commonly used exponential distributions, e.g., the Gaussian distribution as used in the classical KF, to capture the occurrence of an outlier. For instance, heavy-tailed Gaussian-mixture [7; 8] and t-distributed noise models [9] are used to modify the KF for better robustness. In [10], an outlier is viewed as a result of measurement noises with variable covariances. Assuming the noise covariances to follow an inverse Gamma distribution, it proposes to add to the KF a procedure of adaptive identification of the key parameters involved in the inverse Gamma distribution. Methods of the second category seek to assign the measurement sample at each time instant with a weight, in an attempt of downweighting outlying measurements. In [3], an expectation-maximization algorithm is used to enable adaptive determination of the weight for a measurement. The results are generalized in [11] and extended to the smoothing problem. In [12; 13], a measurement-weighting-based prewhitening procedure is designed to decorrelate outliers from normal measurements as a basis for building a robust KF technique. The third category considers simultaneous state and outlier estimation, which regards an outlier as an input to the system's measurement process. The literature includes a few studies based on minimum variance unbiased estimation [14–17]. In addition, Bayesian methods are developed in [18; 19] to achieve joint state and outlier estimation from a probabilistic perspective. Despite the usefulness, these robust KF techniques come with a dramatic increase in computational complexity, due to the needed iterative optimization or other computationally expensive procedures. Besides, they often require redundant measurements in order to either differentiate outliers from normal measurements or estimate them directly. This, however, is not always possible, because a real system often allows only a limited number of sensors to be deployed.

In addition to the above outlier-robust KF methods, one can also find some other types of estimation approaches in the literature capable of suppressing outliers. Among them, a well-known one is $\mathcal{H}_\infty$ filtering [20; 21], which considers outliers as unknown yet bounded uncertainty. However, this approach may introduce significant conservatism as it performs worst-case estimation by design. A stubborn observer is developed in [22], which employs a saturation function in the output injection signal to mitigate the influence of outliers. This method is not only computationally fast but also can deal with large outliers.

In this work, we propose to robustify the classical KF with an innovation saturation mechanism. The innovation plays a key role in correcting the state prediction in the KF but can be distorted by outliers. To overcome such a vulnerability, our mechanism saturates the innovation when it is unreasonably large in order to suppress the effects of outliers. The design of the mechanism includes a procedure for adaptively adjusting the saturation bound to effectively grasp the change of the innovation. Along this line, we develop the saturated KF (SKF) approaches for both continuous- and discrete-time

H. Fang is with Department of Mechanical Engineering, University of Kansas, Lawrence, KS 66045, USA (e-mail: fang@ku.edu) .

M. A. Haile is with Vehicle Technology Directorate, US Army Research Laboratory, Aberdeen, MD 21005, USA (e-mail: mulugeta.a.haile.civ@mail.mil).

Y. Wang is with Mitsubishi Electric Research Laboratories, Cambridge, MA 02139, USA, which funded his work on this topic (e-mail: yebinwang@ieee.org).

systems. It can be proven that the SKF would produce bounded-error estimation when the outliers are bounded in magnitude. Numerical simulation further illustrates the effectiveness. This work is a generalization of [22] to the KF. Compared to methods in the literature, the proposed SKF approaches are structurally concise, computationally efficient, and robust against even very large outliers, thus lending itself well to practical application.

This paper is organized as follows. Section II develops the SKF for linear continuous-time systems and analyzes its stability. Section III extends the results to discrete-time systems. A numerical example is provided in Section IV to illustrate the usefulness of the proposed design. Finally, our concluding remarks are presented in Section V.

Notation: Notations used throughout the paper are standard. The n-dimensional Euclidean space is denoted as $\mathbb{R}^n$. For a vector, $\|\cdot\|$ denotes its 2-norm. The notation I is an identity matrix; $X > 0$ (≥ 0) means that X is a real, symmetric and positive definite (semidefinite) matrix; for a symmetric block matrix, we use a star $(\star)$ to represent a symmetry-induced block; the notation $\mathrm{diag}(\ldots)$ stands for a block-diagonal matrix. The minimum and maxim eigenvalues of a real, symmetric matrix are denoted as $\underline{\lambda}(\cdot)$ and $\bar{\lambda}(\cdot)$. Matrices are assumed to be compatible for algebraic operations if their dimensions are not explicitly stated.

II. SKF FOR CONTINUOUS-TIME SYSTEMS

This section develops the SKF approach for a linear continuous-time system and then analyzes its stability.

A. SKF Design

Consider the following model

$$\begin{cases} \dot{x}_t = Ax_t + w_t, \\ y_t = Cx_t + Dd_t + v_t, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ the measurement vector, and $w_t \in \mathbb{R}^n$ and $v_t \in \mathbb{R}^p$ zero-mean, mutually independent noises with covariances given by $Q \geq 0$ and $R > 0$, respectively. Note that the measurement y_t is subjected to the outlier effects caused by a disturbance $d_t \in \mathbb{R}^m$. As often needed for estimation, we assume that (A, C) is detectable and $(A, Q^{\frac{1}{2}})$ stabilizable.

Modifying the classical KF, we propose the the following state estimation procedure:

$$\dot{\hat{x}}_t = A\hat{x}_t + K_t \cdot \mathrm{sat}_\sigma \left(y_t - C\hat{x}_t \right), \tag{2a}$$

$$K_t = P_t C^\top R^{-1}, \tag{2b}$$

$$\dot{P}_t = AP_t + P_t A^\top + Q - K_t R K_t^\top, \tag{2c}$$

where K_t is the estimation gain matrix, and P_t a positive definite matrix that represents the estimation error covariance. Note that, for the classical KF, the state estimation is corrected by the innovation $(y_t - C\hat{x}_t)$. Its effectiveness, however, can be compromised if y_t is corrupted by an outlier. To overcome this issue, we draw analogy to the design of a stubborn observer in [22] and use a saturated innovation instead, as shown in (2a). Specifically, it is defined as

$$\mathrm{sat}_\sigma \left(y_t - C\hat{x}_t \right) = \begin{bmatrix} \vdots \\ \mathrm{sat}_{\sqrt{\sigma_i}} \left(y_{i,t} - C_i\hat{x}_t \right) \\ \vdots \end{bmatrix}, \tag{3}$$

where $\sigma_i > 0$, y_i is the i-th element of y, and C_i the i-th row of C. For a variable r, the saturation function is defined as $\mathrm{sat}_\epsilon(r) = \max\left\{-\epsilon, \min\{\epsilon, r\}\right\}$. For (3), one can approximately view the saturation range $\left[-\sqrt{\sigma_i}, \sqrt{\sigma_i}\right]$ as an anticipated range of the innovation. If falling within this range, the innovation is considered reasonable and applied without change to update the state estimation. Otherwise, it may be affected by an outlier and thus saturated so as not to mislead the estimation.

It is observed that a fixed saturation bound can be limited in efficacy of rejecting outliers as it may confuse with an outlier a certain measurement generating a large innovation, or vice versa. In addition, it can also be difficult in practice to select a fixed bound, especially when knowledge about possible outliers afflicting a system is scarce. The following procedure is thus introduced to adaptively adjust the saturation bound:

$$\dot{\sigma}_{i,t} = \mu_i \sigma_{i,t} + \gamma_i \left(y_{i,t} - C_i\hat{x}_t \right)^2, \ \sigma_{i,0} > 0, \tag{4}$$

for $i = 1, 2, \ldots, p$, where $\mu_i < 0$ and $\gamma_i > 0$. We define $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_p)$. With (4), σ_i can dynamically change according to the innovation $(y_{i,t} - C_i\hat{x}_t)$. This will enable an improved discernment between an outlier and a normal measurement.

Putting together (2)-(4), we obtain the SKF for the linear continuous-time system in (1). Next, we analyze the its stability properties.

B. Stability Analysis

Define the state estimation error as $e_t = \hat{x}_t - x_t$. The dynamics of e_t is governed by

$$\dot{e}_t = Ae_t - K_t \cdot \mathrm{sat}_\sigma (Ce_t - Dd_t - v_t) - w_t. \tag{5}$$

To proceed further, we define the following matrix

$$S_t = \begin{bmatrix} M_t - \alpha P_t^{-1} & -C^\top \left(R^{-1} + W \right) & C^\top (\Gamma - R^{-1}) D \\ \star & 2W & WD \\ \star & \star & U \end{bmatrix},$$

where $M_t = P_t^{-1} Q P_t^{-1} + C^\top (R^{-1} - \Gamma) C$, W is a diagonal positive definite matrix, U a positive definite matrix, and $\alpha > 0$ a positive scalar. Furthermore, we recall a well-known fact [23]: if $(A, Q^{\frac{1}{2}})$ is stabilizable and (A, C) detectable, P_t for $P_0 \geq 0$ in (2c) will approach a unique positive-definite solution $\bar{P}$ satisfying

$$AP_\infty + P_\infty A^\top + Q - P_\infty C^\top R^{-1} C P_\infty = 0.$$

We also assume $w_t = 0$ and $v_t = 0$ to perform the stability analysis in a deterministic setting. Hence, Q and R would be considered as weight matrices rather than covariances.

The following result is obtained about the stability of the proposed SKF.

Theorem 1: Suppose $w_t = 0$, $v_t = 0$ and $\|d_t\| \leq \delta < \infty$, where $\delta > 0$. If there exist P_0, W, U, α and Γ such that $S(t) \geq 0$ and $0 < \alpha \leq -\max(\mu_i)$ for $i = 1, 2, \ldots, p$, then the estimation error e_t is upper bounded with

$$\|e_t\| \leq \sqrt{\frac{1}{c_2}\left[e^{-\alpha t}V_0 + \frac{1}{\alpha}(1-e^{-\alpha t})c_1\delta^2\right]}, \tag{6a}$$

$$\lim_{t\to\infty} \|e_t\| \leq \sqrt{\frac{c_1}{\alpha c_3}}\delta, \tag{6b}$$

where $c_1 = \bar{\lambda}(U + D^\top \Gamma D)$, $c_2 = \underline{\lambda}(P_t^{-1})$ and $c_3 = \underline{\lambda}(P_\infty^{-1})$.

Proof: We consider using the Lyapunov function

$$V_t = e_t^\top P_t^{-1} e_t + \textstyle\sum_i \sigma_{i,t}.$$

The first-order time derivative of V_t along (5) is

$$\begin{aligned}
\dot{V}_t &= 2e_t^\top P_t^{-1}\dot{e}_t + e_t^\top \frac{\mathrm{d}\left(P_t^{-1}\right)}{\mathrm{d}t} e_t + \sum_i \dot{\sigma}_{i,t} \\
&= 2e_t^\top P_t^{-1}\left[Ae_t - K_t \cdot \mathrm{sat}_\sigma\left(Ce_t - Dd_t\right)\right] \\
&\quad - e_t^\top P_t^{-1}\left(AP_t + P_tA^\top + Q - K_tRK_t^\top\right)P_t^{-1}e_t \\
&\quad + \textstyle\sum_i \mu_i\sigma_{i,t} + (Ce_t - Dd_t)^\top \Gamma (Ce_t - Dd_t) \\
&= -e_t^\top P_t^{-1}QP_t^{-1}e_t + e_t^\top C^\top(R^{-1} + \Gamma)Ce_t \\
&\quad - 2e_t^\top C^\top R^{-1}\mathrm{sat}_\sigma\left(Ce_t - Dd_t\right) - 2e_t^\top C^\top \Gamma Dd_t \\
&\quad + d_t^\top D^\top \Gamma D d_t + \textstyle\sum_i \mu_i\sigma_{i,t}.
\end{aligned}$$

Let us define $s_t = Ce_t - Dd_t - \mathrm{sat}_\sigma(Ce_t - Dd_t)$. Then,

$$\begin{aligned}
\dot{V}_t &= -e_t^\top M_t e_t + 2e_t^\top C^\top R^{-1}s_t + 2e_t^\top C^\top(R^{-1} - \Gamma)Dd_t \\
&\quad + d_t^\top D^\top \Gamma D d_t + \textstyle\sum_i \mu_i\sigma_{i,t}.
\end{aligned}$$

By [24, Lemma 1.6], we have

$$-s_t^\top W\left(s_t - Ce_t + Dd_t\right) \geq 0.$$

It then follows that

$$\begin{aligned}
\dot{V}_t &\leq \dot{V}_t - 2s_t^\top W\left(s_t - Ce_t + Dd_t\right) \\
&= -e_t^\top M_t e_t + 2e_t^\top C^\top\left(R^{-1} + W\right)s_t - 2s_t^\top W s_t \\
&\quad + 2e_t^\top C^\top(R^{-1} - \Gamma)Dd_t - 2s_t^\top W D d_t + \\
&\quad + d_t^\top D^\top \Gamma D d_t + \textstyle\sum_i \mu_i\sigma_{i,t} \\
&= -\begin{bmatrix} e_t \\ s_t \\ d_t \end{bmatrix}^\top \begin{bmatrix} M_t & -C^\top\left(R^{-1} + W\right) & C^\top(\Gamma - R^{-1})D \\ \star & 2W & WD \\ \star & \star & U \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} e_t \\ s_t \\ d_t \end{bmatrix} + d_t^\top\left(U + D^\top\Gamma D\right)d_t + \textstyle\sum_i \mu_i\sigma_{i,t}.
\end{aligned}$$

If $S_t \geq 0$, we have

$$\begin{aligned}
\dot{V}(t) &\leq -\alpha e_t^\top P_t^{-1}e_t - \alpha\textstyle\sum_i \sigma_{i,t} + \sum_i(\mu_i + \alpha)\sigma_{i,t} \\
&\quad + d_t^\top\left(U + D^\top\Gamma D\right)d_t.
\end{aligned}$$

If $0 < \alpha \leq -\max(\mu_i)$, one has $\mu_i + \alpha \leq 0$. Then,

$$\begin{aligned}
\dot{V}_t &\leq -\alpha V_t + d_t^\top\left(U + D^\top\Gamma D\right)d_t \\
&\leq -\alpha V_t + c_1\|d_t\|^2 \\
&\leq -\alpha V_t + c_1\delta^2.
\end{aligned}$$

Hence,

$$V_t \leq e^{-\alpha t}V_0 + \frac{1}{\alpha}(1 - e^{-\alpha t})c_1\delta^2.$$

Furthermore, $V_t \geq c_2\|e_t\|^2$. Then,

$$\|e_t\|^2 \leq \frac{1}{c_2}\left[e^{-\alpha t}V_0 + \frac{1}{\alpha}(1 - e^{-\alpha t})c_1\delta^2\right],$$

which implies (6a). When $t \to \infty$, we can obtain (6b). •

Theorem 1 shows that, if the system is noise-free and the outliers are upper-bounded, the proposed SKF can lead to bounded-error estimation under certain conditions. Meanwhile, it is easy to verify that e_t will exponentially approach zero as $t \to \infty$ if $d_t = 0$.

Remark 1: For the proposed SKF, selection of P_0 is critical to make the condition $S_t \geq 0$ satisfied. Given the structure of S_t, it is cautioned that too large a P_0 may bring the risk of divergent estimation. However, note that P_t is monotonically non-decreasing for $P_0 = 0$ if $(A, Q^{\frac{1}{2}})$ is stabilizable and (A, C) detectable [23]. Leveraging this property, we recommend that P_0 be set to zero or close to zero when the SKF approach is to be implemented. •

III. SKF FOR DISCRETE-TIME SYSTEMS

Extending the notion in Section II, this section investigates the development of SKF for linear discrete-time systems.

A. SKF Design

Consider a discrete-time system

$$\begin{cases} x_{k+1} = Ax_k + w_k, \\ \quad y_k = Cx_k + Dd_k + v_k. \end{cases} \tag{7}$$

The notations in above are the same as in Section II. Still, the noises w_k and v_k are zero-mean, mutually independent with covariances $Q \geq 0$ and $R > 0$, respectively. We also assume that $(A, Q^{\frac{1}{2}})$ is stabilizable and (A, C) detectable without loss of generality. In addition, we assume that A is invertible.

For this system, we propose the SKF as follows:

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1}, \tag{8a}$$

$$P_{k|k-1} = AP_{k-1|k-1}A^\top + Q, \tag{8b}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \cdot \mathrm{sat}_\sigma\left(y_k - C\hat{x}_{k|k-1}\right), \tag{8c}$$

$$K_k = P_{k|k-1}C^\top\left(CP_{k|k-1}C^\top + R\right)^{-1}, \tag{8d}$$

$$P_{k|k} = P_{k|k-1} - K_k\left(CP_{k|k-1}C^\top + R\right)K_k^\top, \tag{8e}$$

where $\hat{x}_{k|k-1}$ is the one-step-forward prediction of x_k, and $\hat{x}_{k|k}$ the updated estimate when y_k arrives to correct the prediction. In addition, K_k is the estimation gain, and $P_{k|k-1}$ and $P_{k|k}$ the estimation error covariances in the standard KF. Akin to (2), an innovation saturation mechanism is used to deal with possible measurement outliers, as shown in (8c). The saturation bound is dynamically adjusted by

$$\sigma_{i,k+1} = \mu_i\sigma_{i,k} + \gamma_i\left(y_{i,k} - C_i\hat{x}_{k|k-1}\right)^2, \ \sigma_{i,0} > 0, \tag{9}$$

for $i = 1, 2, \ldots, p$ where $-1 < \mu_i < 1$ and $\gamma_i > 0$.

B. Stability Analysis

Now, let us consider the stability analysis for the above SKF. Defining the state prediction error as $e_k = \hat{x}_{k|k-1} - x_k$, its dynamics can be expressed as

$$
\begin{aligned}
e_{k+1} &= \hat{x}_{k+1|k} - x_{k+1} \\
&= Ae_k - AK_k \cdot \mathrm{sat}_\sigma \left(Ce_k - Dd_k - v_k \right) - w_k. \quad (10)
\end{aligned}
$$

Before proceeding further, we show some results that will be needed later. Given that $(A, Q^{\frac{1}{2}})$ is stabilizable and that (A, C) detectable, $P_{k|k-1}$ will converge to a fixed positive definite matrix P_∞ satisfying

$$
\begin{aligned}
P_\infty = AP_\infty A^\top + Q - AP_\infty C^\top \left(CP_\infty C^\top + R \right)^{-1} \\
\cdot CP_\infty A^\top.
\end{aligned}
$$

It is also known that $P_{k|k-1}$ is upper and lower bounded, and so is $P_{k|k}$. Hence, there should exist an $\epsilon > 0$ such that $P_{k|k}^{-1} \leq \epsilon I$. Then,

$$
\begin{aligned}
&P_{k+1|k}^{-1} = \left(AP_{k|k} A^\top + Q \right)^{-1} \\
&= A^{-\top} \left(P_{k|k} + A^{-1} Q A^{-\top} \right)^{-1} A^{-1} \\
&= A^{-\top} \left[P_{k|k}^{-1} - P_{k|k}^{-1} \left(P_{k|k}^{-1} + A^\top Q^{-1} A \right)^{-1} P_{k|k}^{-1} \right] A^{-1} \\
&\leq A^{-\top} \left[P_{k|k}^{-1} - P_{k|k}^{-1} \left(\epsilon I + A^\top Q^{-1} A \right)^{-1} P_{k|k}^{-1} \right] A^{-1} \\
&= A^{-\top} \left[P_{k|k}^{-1} - P_{k|k}^{-1} \bar{Q} P_{k|k}^{-1} \right] A^{-1},
\end{aligned}
$$

where $\bar{Q} = \left(\epsilon I + A^\top Q^{-1} A \right)^{-1}$. We also define

$$
Z_k = \begin{bmatrix} T_{1,k} - \alpha P_{k|k-1}^{-1} & T_{2,k} - C^\top W & T_{3,k} \\ \star & T_{4,k} + 2W & T_{5,k} + WD \\ \star & \star & U \end{bmatrix},
$$

where W is a diagonal positive definite matrix, U a positive definite matrix, $\alpha > 0$ a positive scalar, and

$$
\begin{aligned}
T_{1,k} &= C^\top R^{-1} C + P_{k|k}^{-1} \bar{Q} P_{k|k}^{-1} - P_{k|k}^{-1} \bar{Q} C^\top R^{-1} C \\
&\quad - C^\top R^{-1} C \bar{Q} P_{k|k}^{-1} - C^\top R^{-1} C \left(P_{k|k} - \bar{Q} \right) \\
&\quad \cdot C^\top R^{-1} C - C^\top \Gamma C, \\
T_{2,k} &= -C^\top R^{-1} + P_{k|k}^{-1} \bar{Q} C^\top R^{-1} \\
&\quad + C^\top R^{-1} C \left(P_{k|k} - \bar{Q} \right) C^\top R^{-1}, \\
T_{3,k} &= \Big[- C^\top R^{-1} + P_{k|k}^{-1} \bar{Q} C^\top R^{-1} \\
&\quad + C^\top R^{-1} C \left(P_{k|k} - \bar{Q} \right) C^\top R^{-1} + C^\top \Gamma \Big] D, \\
T_{4,k} &= -R^{-1} C \left(P_{k|k} - \bar{Q} \right) C^\top R^{-1}, \\
T_{5,k} &= - \left[R^{-1} C \left(P_{k|k} - \bar{Q} \right) C^\top R^{-1} \right] D.
\end{aligned}
$$

In addition, we consider another matrix

$$
T_{6,k} = D^\top \left[R^{-1} C \left(P_{k|k} - \bar{Q} \right) C^\top R^{-1} + \Gamma \right] D.
$$

Here, the upper boundedness of $P_{k|k}$ implies that $T_{6,k}$ is also upper bounded, with the bound denoted as $\bar{T}_6$.

The following theorem shows the result about the stability of the prediction error dynamics.

Theorem 2: Suppose $w_k = 0$, $v_k = 0$, $\|d_k\| \leq \delta < \infty$ and that A is invertible. If there exist $P_{0|-1}$, W, U, α and Γ such that $Z_k \geq 0$ and $0 < \alpha \leq 1 - \max(\mu_i)$ for $i = 1, 2, \ldots, p$, then the estimation error e_k is upper bounded with

$$
\|e_k\| \leq \sqrt{\frac{1}{c_2} \left[(1-\alpha)^k V_0 + \frac{1 - (1-\alpha)^{k-1}}{\alpha} c_1 \delta^2 \right]}, \quad (11a)
$$

$$
\lim_{k \to \infty} \|e_k\| \leq \sqrt{\frac{c_1}{\alpha c_3}} \delta, \quad (11b)
$$

where $c_1 = \bar{\lambda}(\bar{T}_6 + U)$, $c_2 = \underline{\lambda}(P_{k|k-1}^{-1})$, and $c_3 = \underline{\lambda}(P_\infty^{-1})$.

Proof: Consider a Lyapunov function

$$
V_k = e_k^\top P_{k|k-1}^{-1} e_k + \textstyle\sum_i \sigma_{i,k}.
$$

Using (10), we have

$$
\begin{aligned}
&V_{k+1} = e_{k+1}^\top P_{k+1|k}^{-1} e_{k+1} + \textstyle\sum_i \sigma_{i,k+1} \\
&\leq \left[Ae_k - AK_k \cdot \mathrm{sat}_\sigma (CAe_k - Dd_k) \right]^\top \\
&\quad \cdot A^{-\top} \left[P_{k|k}^{-1} - P_{k|k}^{-1} \bar{Q} P_{k|k}^{-1} \right] A^{-1} \\
&\quad \cdot \left[Ae_k - AK_k \cdot \mathrm{sat}_\sigma (Ce_k - Dd_k) \right] \\
&\quad + \textstyle\sum_i \mu_i \sigma_{i,k} + \left(Ce_k - Dd_k \right)^\top \Gamma \left(Ce_k - Dd_k \right) \\
&= e_k^\top \left[P_{k|k}^{-1} - P_{k|k}^{-1} \bar{Q} P_{k|k}^{-1} \right] e_k - 2e_k^\top \left[P_{k|k}^{-1} - P_{k|k}^{-1} \bar{Q} P_{k|k}^{-1} \right] \\
&\quad \cdot K_k \cdot \mathrm{sat}_\sigma (Ce_k - Dd_k) + \mathrm{sat}_\sigma^\top (Ce_k - Dd_k) \cdot K_k^\top \\
&\quad \cdot \left[P_{k|k}^{-1} - P_{k|k}^{-1} \bar{Q} P_{k|k}^{-1} \right] K_k \cdot \mathrm{sat}_\sigma (Ce_k - Dd_k) \\
&\quad + \left(Ce_k - Dd_k \right)^\top \Gamma \left(Ce_k - Dd_k \right) + \textstyle\sum_i \mu_i \sigma_{i,k}.
\end{aligned}
$$

Let us define

$$
s_k = Ce_k - Dd_k - \mathrm{sat}_\sigma (Ce_k - Dd_k).
$$

In addition, we have $P_{k|k}^{-1} = P_{k|k-1}^{-1} + C^\top R^{-1} C$, $P_{k|k}^{-1} K_k = C^\top R^{-1}$ and $K_k^\top P_{k|k}^{-1} K_k = R^{-1} C P_{k|k} C^\top R^{-1}$. These relations can be readily proven. It then follows that

$$
\begin{aligned}
V_{k+1} &\leq e_k^\top P_{k|k-1}^{-1} e_k - e_k^\top T_{1,k} e_k - 2e_k^\top T_2 s_k - 2e_k^\top T_{3,k} d_k \\
&\quad - s_k^\top T_{4,k} s_k - 2s_k^\top T_{5,k} d_k + d_k^\top T_{6,k} d_k + \textstyle\sum_i \mu_i \sigma_{i,k}.
\end{aligned}
$$

According to [24, Lemma 1.6], we have

$$
-s_k^\top W (s_k - Ce_k + Dd_k) \geq 0.
$$

It can be obtained that

$$
\begin{aligned}
V_{k+1} &\leq V_{k+1} - 2s_k^\top W (s_k - Ce_k + Dd_k) \\
&= e_k^\top P_{k|k-1}^{-1} e_k - e_k^\top T_{1,k} e_k - 2e_k^\top (T_2 - C^\top W) s_k \\
&\quad - 2e_k^\top T_{3,k} d_k - s_k^\top \left(T_{4,k} + 2W \right) s_k \\
&\quad - 2s_k^\top \left(T_{5,k} + WD \right) d_k + d_k^\top T_{6,k} d_k + \textstyle\sum_i \mu_i \sigma_{i,k} \\
&= e_k^\top P_{k|k-1}^{-1} e_k - \begin{bmatrix} e_k \\ s_k \\ d_k \end{bmatrix}^\top \\
&\quad \cdot \begin{bmatrix} T_{1,k} & T_{2,k} - C^\top W & T_{3,k} \\ \star & T_{4,k} + 2W & T_{5,k} + WD \\ \star & \star & U \end{bmatrix} \begin{bmatrix} e_k \\ s_k \\ d_k \end{bmatrix} \\
&\quad + d_k^\top \left(T_{6,k} + U \right) d_k + \textstyle\sum_i \mu_i \sigma_{i,k}.
\end{aligned}
$$

If $Z_k \geq 0$, then

$$V_{k+1} \leq (1-\alpha) e_k^\top P_{k|k-1}^{-1} e_k + (1-\alpha) \sum_i \sigma_{i,k} + \sum_i (\mu_i + \alpha - 1)\sigma_{i,k} + d_k^\top \left(T_{6,k} + U\right) d_k.$$

Because $0 < \alpha \leq 1 - \max(\mu_i)$ for $i = 1, 2, \ldots, p$,

$$\begin{aligned} V_{k+1} &\leq (1-\alpha) V_k + d_k^\top \left(T_{6,k} + U\right) d_k \\ &\leq (1-\alpha) V_k + \bar{\lambda} \left(\bar{T}_6 + U\right) \delta^2, \end{aligned}$$

from which one can easily obtain (11a)-(11b). •

Remark 2: For the discrete-time case, it is also recommended that $P_{k|k-1}$ is initialized with a small $P_{0|-1}$, which can be zero or near zero, in order to make the conditions in Theorem 2 satisfied more easily. •

Remark 3: It is observed that stability analysis of the SKF approach for both continuous- and discrete-time systems involves simplifying assumptions, e.g., zero process and measurement noises and invertible A (for the discrete-time case). The zero-noise assumption allows us to perform the analysis in a deterministic setting. This move makes the problem more tractable by avoiding the nonlinear, saturation-function-based transformation of random noise variables, which will appear otherwise in a stochastic setting and may obstruct a proof. Yet, this does not imply that the SKF approach can only be applied to deterministic systems. From a large number of simulations, we still consistently observe satisfactory estimation performance when noises exist. An example in this respect is also shown in Section IV. Besides, although the assumption that A of a discrete-time system is invertible sounds strong, it is noticed that a zero-order-hold discretization of a linear continuous-time system will always result in an invertible A. How to relax these assumptions still presents interesting challenges, which will motivate our further work. •

Remark 4: It is worth pointing out that different strategies can be developed and used to enable the saturation bound adjustment. For instance, one can modify (4) as

$$\dot{\sigma}_{i,t} = \mu_i \sigma_{i,t} + \gamma_i |y_{i,t} - C_i \hat{x}_t|,$$

or (9) as

$$\sigma_{i,k+1} = \mu_i \sigma_{i,k} + \gamma_i |y_{i,k} - C_i \hat{x}_{k|k-1}|,$$

which should come with a different set of stability conditions. It will hence be interesting to investigate different methods for adjusting the saturation bound and assessing their performance. •

IV. Illustrative Example

In this section, we present a numerical simulation example to illustrate the effectiveness of the proposed SKF for discrete-time systems. Consider a discrete-time system as in (7) with

$$A = \begin{bmatrix} 0.8 & 0.32 & 0 \\ 0 & -0.67 & 0.5 \\ 0.2 & -0.1 & -0.9 \end{bmatrix}, \; B = \begin{bmatrix} 0.4 \\ -1.2 \\ 0.9 \end{bmatrix},$$
$$C = \begin{bmatrix} 1.4 & 0.6 & 0.3 \end{bmatrix}, \; D = 1.$$

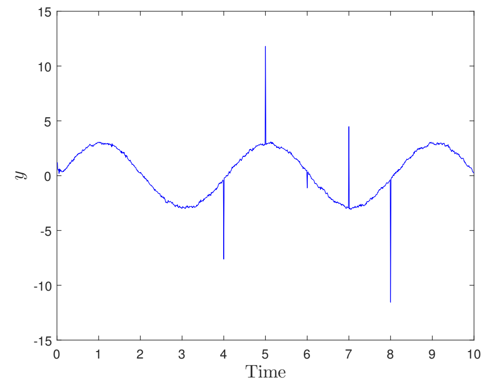


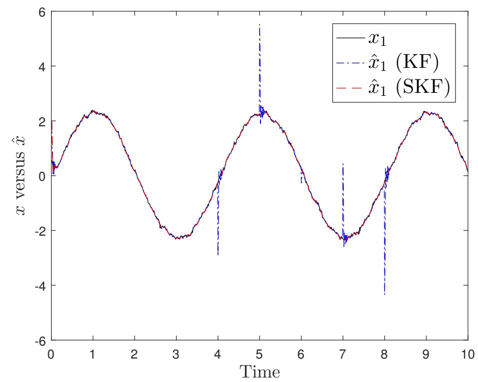
Fig. 1: Output measurement profile affected by outliers.

Suppose that the system is excited by a sinusoidal input $u_k = 2\sin(k\pi T_s/2)$ with the sampling period $T_s = 0.01$ seconds. The noise covariances are given by $Q = 10^{-3} I$ and $R = 10^{-3}$, respectively. In addition, the measurement y_k is subjected to occasional outliers, d_k, at the fourth, fifth, sixth, seventh and eighth seconds, with a magnitude of -7.25, 9, -1.5, 7.5 and -11.25. The profile of y_k is depicted in Fig. 1, which shows that the outlier measurements considerably deviate from the norm.

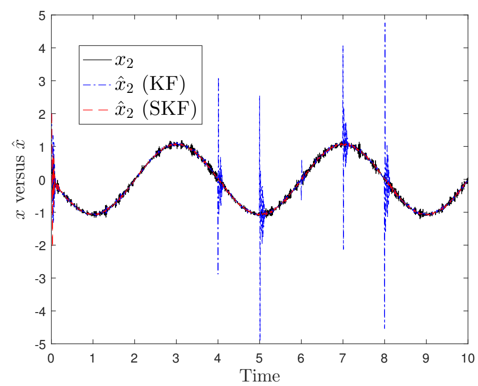
Both the classical KF and the proposed SKF are applied to perform estimation for the considered system for a comparison. For the SKF, $\mu = 0.7$ and $\gamma = 1$. The initial $P_{0|-1}$ is set as $P_{0|-1} = 0.1I$ for both filters. The estimation results are summarized in Fig. 2. It is observed from Fig. 2(a)-2(c) that the classical KF leads to satisfactory estimation for each state variable when there is no outlier. However, inaccurate and quite turbulent estimation results when an outlier arises, implying risks for practical system monitoring or control. For the SKF, it demonstrates estimation performance comparable to the classical KF when the system is running free of outliers. When outliers appear, it can still maintain a smooth and accurate estimation, thanks to the innovation saturation. This reflects a remarkable effectiveness of the SKF in suppressing the undesirable influence of outliers. The profile of σ is illustrated in Fig. 2(d), which shows a dynamic change in response to the appearance of outliers.

V. Conclusion

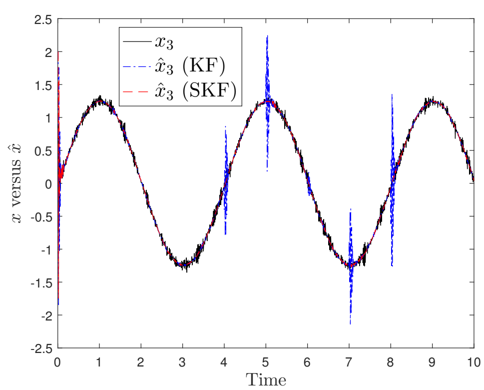
The KF has gained wide application in numerous fields as one of the most popular state estimation tools. Its performance, however, can be reduced by measurement outliers due to sensor anomaly, data transmission errors or cyber attacks. Aiming to enhance robustness of the KF, this paper proposes to apply an innovation saturation mechanism to the standard KF. This mechanism saturates the innovation process, which is crucial for correcting the state estimation, to ensure that a reasonable correction is applied when outliers occur. The design is performed for both continuous- and discrete-time Kalman filtering. Theoretical analysis provides useful stability properties of the proposed approaches. Numerical



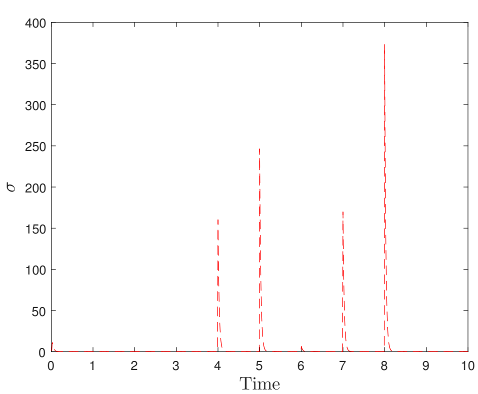
(a)



(b)



(c)



(d)

Fig. 2: Estimation of the state variables: (a) x_1 versus $\hat{x}_1$; (b) x_2 versus $\hat{x}_2$; (c) x_3 versus $\hat{x}_3$; change of σ.

simulation further illustrates that the proposed design can bring about significant robustness against outliers.

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