Reduced Complexity Control Design for Symmetric LPV Systems

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I. INTRODUCTION

Linear parameter varying (LPV) systems have been used in multiple contexts, such as gain scheduling, robust control, and control of periodic systems [1]. The recent survey [2] discusses the impact of several methods for controlling LPV systems with applications in different domains. A subclass of LPV systems are polytopic LPV systems [3], [4]. In polytopic LPV systems, the dynamics belong to the convex-hull of a finite set of extreme-dynamics, often called the extreme systems. The trajectories of a polytopic LPV system can also be obtained as trajectories of a polytopic difference inclusion with the same extreme systems [5].

A method for designing parameter-dependent controllers for discrete-time polytopic LPV systems based on polytopic parameter-dependent Lyapunov was presented in [6]. Parameter-dependent Lyapunov functions (pLF) were first introduced in [7] as a tool for robust stability analysis of uncertain autonomous systems. A pLF is the convex combination of Lyapunov functions for the extreme dynamics. Stability is guaranteed by ensuring that the parameter-dependent Lyapunov function value decreases regardless of the evolution of the LPV parameter. Parameter-dependent Lyapunov functions are a less conservative alternative to using a common Lyapunov function for all the extreme systems. By considering functions that depend on the system parameter, pLFs provide less conservative stability certification.

In [7] it was shown how pLFs could be used for designing a robust linear controller for uncertain non-autonomous systems modeled as polytopic LPVs. As opposed to [7], in [6] the parameter of the LPV system at the current time is assumed to be known. A parameter-dependent controller can be designed, where the controller gains depend on the current LPV system parameter. Hence the controller gain adjusts as the LPV dynamics change. A parameter-dependent Lyapunov function is used to certify the stability of the LPV system in closed-loop with the parameter-dependent controller. More recently in [8] this method was extended to design output feedback controllers, where both the estimator and the controller are designed based on parameter-dependent Lyapunov functions. The methods in [7] and [6] have been extended to model predictive control (MPC), respectively for robust MPC [9] and MPC of LPV systems [10].

The design methods for parameter-dependent controllers and Lyapunov functions presented in [6]–[10] require the solution of linear matrix inequalities (LMIs) [5]. The size of the LMIs is proportional to the dimension of the system times the number of extreme systems. The multiple depends on the specific design method and objectives. The LMIs can become very large when the pLFs and parameter-dependent controller are designed simultaneously for LPV systems with many extreme systems. Thus designing parameter-dependent controllers and Lyapunov functions can be a challenging problem. This is especially true in applications to MPC [9], [10], where the LMI are solved in real time during controller execution.

In this paper we propose a method for reducing the number and dimension of the design variables and the number of linear matrix inequalities using symmetry. For linear parameter varying systems, symmetries are transformations of the states and inputs that do not change the set of systems. Symmetries relate the extreme-dynamics of the LPV systems. These relationships can be exploited to reduce the number of extreme-controllers and extreme-Lyapunov functions that must be designed. In addition these symmetric relationships can be used to reduce the number of linear matrix inequalities that the design variables must satisfy.

Symmetry has a long history of being used to analyze and simplify control design [11]. In [12] it was shown that controllability and stability in large scale systems can be determined by using symmetry to decompose a system and then check the smaller decoupled subsystems. In [13] symmetry adapted basis was used to simplify the design of \( H_2 \) and \( H_\infty \) controllers. In [14] symmetry was used to simplify the process of finding the transition probabilities of Markov-Chains which produce the fastest convergence. In [15] symmetry was used to simplify explicit model predictive controllers.

This paper is organized as follows. In Section II we define linear parameter varying systems, and parameter-dependent controllers and Lyapunov functions. We summarize a linear matrix inequality based procedure for designing parameter-dependent controllers and Lyapunov functions. In Section III we define symmetry for linear parameter varying systems, and parameter-dependent controllers and Lyapunov functions. In Section IV we provide three complementary methods for reducing the complexity of the controller and Lyapunov function design. In Section V we apply our reduced complexity control design to building control. We show that our method reduces the number of decision variables and linear matrix inequalities from a combinatorial function of the number of rooms to a polynomial function. Proofs have been omitted for space.

A. Mathematical Background on Group Theory

In this section we review the relevant concepts, notation, and results from group theory [16], [17]. Permutations will be described using the Cayley notation i.e. the permutation \( \pi \) such that \( \pi(1) = 2 \), \( \pi(2) = 3 \), and \( \pi(3) = 1 \) will be denoted as \( \pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \). A group \( (G, \circ) \) is a set \( G \) along with a binary operation \( \circ \) such that the operator is associative, the set \( G \) is closed under the operation \( \circ \), the set includes an identity element and the inverse of each element. In matrix groups the operator \( \circ \) is matrix multiplication and the identity element is the identity matrix \( I \). In permutation groups
the operator $\circ$ is function composition and the identity element is the identity permutation $e$. A group that contains only the identity element is called trivial. For notational simplicity we will drop the $\circ$ and write $gh$ for $g \circ h$.

A group $G$ acts on a set $X$ if its elements $g \in G$ index functions $f_g : X \to X$ satisfying the group law: $f_g \circ f_h = f_{gh}$, where $\circ$ is function composition. A matrix group is defined in terms of its action $\Theta_g \in \mathbb{R}^{n \times n}$ on a vector-space $\mathbb{R}^n$ and a permutation group is defined by its action $\pi_g$ on an index set $I \subseteq \mathbb{N}$.

The orbit $G(i) = \{\pi_g(i) : g \in G\} \subseteq I$ is the image $\pi_g(i)$ of the index $i \in I$ under every permutation $\pi_g$ for $g \in G$ in the group $G$. We will denote the set of orbits $G(i)$ by $I/G = \{G(i_1), \ldots, G(i_r)\}$, read as $I$ modulo $G$, where $i_1, \ldots, i_r$ is a set that contains one representative $i_j$ from each orbit $G(i_j) \subseteq I$. With abuse of notation we will equate the set of orbits $I/G$ with a set of representative indices $I/G = \{i_1, \ldots, i_r\}$. The orbits $G(i)$ for $i \in I/G$ partition the index set $I$ into disjoint equivalence classes $I = \bigcup_{i \in I/G} G(i)$.

The stabilizer subgroup $G_i = \{g \in G : \pi_g(i) = i\}$ is the subset of group elements $g \in G$ that map $i \in I$ to itself $\pi_g(i) = i$. The size of the orbit $G(i)$ and stabilizer $G_i$ are related by the Orbit-Stabilizer theorem which states $|G| = |G(i)||G_i|$ where $| \cdot |$ denotes the cardinality of a set.

The orbits $G(i,j) = \{(\pi_g(i),\pi_j(j)) \in I^2 : g \in G\} \subseteq I^2$ are orbits of pairs $(i,j)$ in $I^2$ of indices $i,j \in I$. We denote the set of orbits by $I^2/G = \{G((i_1,j_1), \ldots, G((i_r,j_r))\}$ read as $I \times I$ modulo $G$ where $\{(i_1,j_1), \ldots, (i_r,j_r)\}$ is a set that contains one representative pair $(i_k,j_k) \in I^2$ from each orbit $G((i_k,j_k)) \subseteq I^2$. With a small abuse of notation we equate the set of orbits $I^2/G$ with a set of representative pairs $I^2/G = \{(i_1,j_1), \ldots, (i_r,j_r)\}$.

The orbits $G(i,j)$ partition the set $I^2$ into disjoint equivalence classes $I^2 = \bigcup_{(i,j) \in I^2/G} G((i,j))$.

Let $\Theta_g \in \mathbb{R}^{n \times n}$ and $\Omega_g \in \mathbb{R}^{m \times n}$ be matrix representations of the group $G$. The commutator $G(\Theta, \Omega) = \{M \in \mathbb{R}^{n \times m} : \Theta_g M = M \Theta_g, \forall g \in G\}$ is the set of all matrices $M \in \mathbb{R}^{n \times m}$ that commute $\Theta_g M = M \Theta_g$ with $\Theta_g$ and $\Omega_g$ for all $g \in G$. The commutator $G(\Theta, \Omega)$ is a subspace of the vector-space $\mathbb{R}^{n \times m}$. This subspace $G(\Theta, \Omega)$ can be described as the span of basis-matrices $M_i \in \mathbb{R}^{n \times m}$ for $i = 1, \ldots, d$ where $d \leq nm$ is the dimension of the vector-space $\mathbb{R}^{n \times m}$. These basis-matrices can be found using techniques from representation theory [17], [18].

II. LINEAR PARAMETER VARYING SYSTEMS

In this section we define linear parameter varying (LPV) systems, parameter-dependent controllers, and parameter-dependent Lyapunov functions. We review a procedure for designing parameter-dependent controllers and Lyapunov functions presented in the literature [6], [8], [10], [19] for synthesizing parameter-dependent controllers and Lyapunov functions.

A. Linear Parameter Varying Systems

In this paper we study the following polytopic linear parameter varying system

$$x(t + 1) = A(\xi)x(t) + Bu(t)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and the parameter varying state-update matrix $A(\xi) = \sum_{i=1}^r \xi_i A_i$ lies inside the polytope $A = \{\sum_{i=1}^r \xi_i A_i : \xi \geq 0, \sum_{i=1}^r \xi_i = 1\}$. We assume this is the minimal representation of $A$. The matrices $A_i \in \mathbb{R}^{n \times n}$ are called the extreme state-update matrices of $A \subseteq \mathbb{R}^{n \times n}$ are indexed by $I = \{1, \ldots, r\}$. We assume the time-varying parameter $\xi$ is known at the current time $t$ and lies in the simplex $\Xi = \{\xi \in \mathbb{R}^r : \xi \geq 0, \sum_{i=1}^r \xi_i = 1\}$.

Our objective is to design a controller that stabilizes the system (1) for any value of the time-varying parameter $\xi$. We use a parameter-dependent control law

$$u(t) = K(\xi)x(t)$$

where the parameter varying controller gain $K(\xi) = \sum_{i=1}^r \xi_i K_i \in \mathbb{R}^{m \times n}$ lies inside the polytope $K = \{\sum_{i=1}^r \xi_i K_i : \xi \in \Xi\}$. The gains $K_i \in \mathbb{R}^{m \times n}$ are called the extreme controller gains of $K \subseteq \mathbb{R}^{m \times n}$ are indexed by $I$.

The stability of the LPV system (1) in closed-loop with the parameter-dependent controller (2) can be certified using a parameter-dependent Lyapunov function

$$V(x, \xi) = x^T P(\xi) x$$

where there exists scalars $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \|x\|^2 \leq x^T P(\xi) x \leq \alpha_2 \|x\|^2$$

for all $\xi \in \Xi$. The parameter varying Lyapunov matrix $P(\xi) = \sum_{i=1}^r \xi_i P_i \in \mathbb{R}^{n \times n}$ lies inside the polytope $P = \{\sum_{i=1}^r \xi_i P_i : \xi \in \Xi\}$. The matrices $P_i = P_i^T \succ 0 \in \mathbb{R}^{n \times n}$ are called the extreme Lyapunov matrices of $P \subseteq \mathbb{R}^{n \times n}$ are indexed by $I$. The closed-loop system is asymptotically stable if

$$(A(\xi) + BK(\xi))^T P(\xi^+) (A(\xi) + BK(\xi)) - P(\xi) < 0$$

for all $\xi^+ \in \Xi$ where $\xi^+ = \xi(t + 1)$. This condition ensures $V(x(t + 1), \xi(t + 1)) - V(x(t), \xi(t)) < 0$ for all $x(t) \neq 0$ and $\xi(t), \xi(t + 1) \in \Xi$.

B. Design of parameter-dependent Controllers and Lyapunov Functions

In this section we review the process presented in [6], [9], [10], [19] for synthesizing parameter-dependent controllers and Lyapunov functions.

The extreme controller gains $K_i = E_i G_i^{-1}$ are parameterized by design variables $E_i \in \mathbb{R}^{m \times n}$ and $G_i \in \mathbb{R}^{n \times n}$ for $i \in I$. The extreme Lyapunov matrices $P_i = S_i^{-1}$ are parameterized by design variables $S_i = S_i^T \in \mathbb{R}^{n \times n}$ for $i \in I$. The design variables $E_i$, $G_i$, and $S_i$ are obtained by solving the following linear matrix inequalities

$$\begin{bmatrix}
G_i + G_i^{-1} - S_i & (A_i G_i + B E_i)^T & G_i^{-1} \\
(A_i G_i + B E_i) & S_i & 0 \\
G_i & 0 & Q_i^{-1} \\
E_i & 0 & 0 & R_i^{-1}
\end{bmatrix} \succeq 0$$

for $i, j \in I$ where $\text{LMI}(G_i, E_i, S_i, S_j)$ is short-hand for the compound matrix above. The matrices $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{n \times n}$ are design variables, similar to the penalty matrices in LQR, used to shape the controller performance. In [10], [19] it was shown that the parameter-dependent controller $K_i = E_i G_i^{-1}$ and Lyapunov function $P_i = S_i^{-1}$ satisfy

$$(A(\xi) + BK(\xi))^T P(\xi^+) (A(\xi) + BK(\xi)) - P(\xi) \leq -Q - K^T R K$$

for all $\xi, \xi^+ \in \Xi$. This condition bounds the decrease of the Lyapunov function.

In [9], [10] similar LMI’s are solved online for robust model predictive control. In model adjustable predictive control [19] the LMI (6) is used to compute the terminal-cost offline. In order to guarantee stability, the decrease in the terminal-cost $x^T P(\xi) x$ for all $\xi \in \Xi$ must be greater than the stage-cost $x^T Q x + u^T R u$. Thus condition (7) guarantees closed-loop stability of the model predictive controller.
III. SYMMETRIC LINEAR PARAMETER VARYING SYSTEMS

In this section, we define symmetry for LPV systems, and parameter-dependent controllers and Lyapunov functions. Then we establish basic results on the existence of symmetric parameter-dependent controllers and Lyapunov functions.

A system of a linear parameter varying system is a state-space $\Theta \in \mathbb{R}^{n \times n}$ and input-space $\Omega \in \mathbb{R}^{m \times m}$ that preserves the dynamics (1)

**Definition 1:** The pair of invertible matrices $(\Theta, \Omega)$ is a symmetry of the polytopic linear parameter varying system (1) if

\[
\Theta A = A \Theta, \quad \Theta B = B \Theta. \tag{8a}
\]

Condition (8a) implies that for each extreme state-update matrix $A_i$ of $A$ there exists another extreme state-update matrix $A_j$ in $A$ such that $\Theta^{-1} A_i \Theta = A_j$. Definition 1 says that the symmetry $(\Theta, \Omega)$ maps the extreme dynamics $x(t+1) = A_i x(t) + B u(t)$ for $i \in I$ to the extreme dynamics $x(t+1) = \Theta A_i \Theta^{-1} x(t) + \Theta B \Omega^{-1} u(t) = A_j x(t) + B u(t)$ for $j \in I$. The set of all symmetries $(\Theta, \Omega)$ of the LPV system (1) forms an infinite group which we denote by $\text{Aut}(A, B)$. To see that $\text{Aut}(A, B)$ is infinite, note that $(\Theta, \Omega) = (\alpha I_n, \alpha I_m)$ is a symmetry for any $\alpha \in \mathbb{R} \setminus \{0\}$. Here we are only interested in symmetries $(\Theta, \Omega)$ that permute the extreme state-update matrices $A_i$ of the set $A$ i.e. $\Theta A_1 \Theta^{-1} = A_j$ for some $i \neq j \in I$. These symmetries form a finite subgroup of $\text{Aut}(A, B)$ which we will denote by $\text{Perm}(A, B)$.

**Proposition 1:** The set $\text{Perm}(A, B)$ of all symmetries $(\Theta, \Omega)$ which permute the extreme-points of $A$ is a finite group.

**Proof:** First we show that $\text{Aut}(A, B)$ under matrix multiplication is a group. Note that matrix multiplication is associative.

1. Identity: $(I_n, I_m) \in \text{Aut}(A, B)$ since $I_n A = A I_n$ and $I_m B = B I_m$.
2. Inverse: Let $(\Theta, \Omega) \in \text{Aut}(A, B)$. Then $(\Theta^{-1}, \Omega^{-1}) \in \text{Aut}(A, B)$ since

   \[
   \Theta A = A \Theta \iff A \Theta^{-1} \iff \Theta^{-1} A, \quad \Theta B = B \Theta \iff B \Omega^{-1} \iff \Theta^{-1} B.
   \]
3. Closure: Let $(\Theta_1, \Omega_1) \in \text{Aut}(A, B)$ and $(\Theta_2, \Omega_2) \in \text{Aut}(A, B)$ since

   \[
   \Theta_1 \Theta_2 A = \Theta_1 A \Theta_2 = \Theta_2 A \Theta_1, \quad \Theta_1 \Theta_2 B = \Theta_1 B \Omega_2 = B \Omega_2 \Theta_2.
   \]

Next, we show that $\text{Perm}(A, B)$ is a finite group. Consider the group $\mathfrak{S}_I$ of permutations on the index set $I$ of extreme state-update matrices $A_i \in A$. By definition of $\text{Aut}(A, B)$ there exists a group homomorphism $\varphi : \text{Aut}(A, B) \rightarrow \mathfrak{S}_I$ from symmetries $(\Theta, \Omega) \in \text{Aut}(A, B)$ of the LPV system (1) to permutations $\pi : I \rightarrow I$ of the extreme state-update matrices $A_i \in A$. The set $\text{Perm}(A, B) = \text{Aut}(A, B) / \text{Ker}(\varphi)$ is the inverse image of this group homomorphism. Therefore $\text{Perm}(A, B)$ is a group. It is finite since $\text{Perm}(A, B) \cong \varphi(\text{Aut}(A, B)) \subseteq \mathfrak{S}_I$ is finite.

We will index the elements $(\Theta_1, \Omega_1)$ of the group $G = \text{Perm}(A, B)$ by $g \in G$. Since $\text{Perm}(A, B)$ is finite there exists a basis for the state-space $\mathbb{R}^n$ and input-space $\mathbb{R}^m$ such that $\Theta_1$ and $\Omega_1$ are orthogonal for all $g \in G$. [20]. For the remainder of this paper, we will assume $\Theta_1$ and $\Omega_1$ are orthogonal matrices $\Theta_1^{-1} = \Theta_1^T$ and $\Omega_1^{-1} = \Omega_1^T$.

We define a permutation action $\pi_g : I \rightarrow I$ for the group $G = \text{Perm}(A, B)$ acting on the index set $I$ of the extreme state-update matrices $A_i \in A$ according to the relation $\pi_g(i) = j$ if

\[
\Theta_y A_i \Theta_y^{-1} = A_j, \quad \text{for all } g \in G. \quad \text{The orbits } G(i) = \{ \pi_g(i) \mid i \in I \mid g \in G \} \text{ of the group } G = \text{Perm}(A, B) \text{ partition the extreme state-update matrices } A_i \text{ into disjoint equivalence classes. For any extreme state-update matrix } A_j \text{ with } j \in G(i) \text{ there exists a state-space transformation } \Theta_g \text{ for some } g \in G \text{ that maps } A_i \text{ to } A_j = \Theta_g A_i \Theta_g^{-1}. \text{ By transitivity, we can find a state-space transformation } \Theta_g \text{ for some } g \in G \text{ that relates any pair of extreme state-update matrices } A_i \text{ and } A_k \text{ with } j, k \in G(i).\]

Example 1 demonstrates the concept of symmetry for linear parameter varying systems.

**Example 1:** Consider a linear parameter varying system of the form (1) with three extreme state-update matrices

\[
A_1 = \begin{bmatrix} 0.9 & 0.1 \\ 0 & 0.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.8 & 0 \\ 0 & 1.0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.9 & 0.1 \\ 0 & 0.8 \end{bmatrix}
\]

and $B \in \mathbb{R}^{2 \times 2}$. Thus $A = \text{conv}(A_1, A_2, A_3)$ and $I = \{1, 2, 3\}$. The vector-fields $f_1(x) = A_1 x$ for the state-update matrices $A_i$ are shown in Figure 1.

This LPV system has one non-trivial symmetry $\Theta_{g_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with $\Omega_{g_1} = \Theta_{g_1}$ satisfying $B \Theta_{g_1} B^T = B$, since $B = I$. Thus the symmetry group is $G = \{g_0, g_1\}$ where $\Theta_{g_0} = \Omega_{g_0} = I \in \mathbb{R}^{2 \times 2}$ is the identity matrix.

The state-space transformation $\Theta_{g_1}$ maps the state-update matrix $A_1$ to $A_2 = A_1 \Theta_{g_1}$ and $A_2$ to $A_3 = \Theta_{g_1} A_3 \Theta_{g_1}^{-1}$. This can be seen in the vector-fields $f_1(x) = A_1 x$ shown in Figure 1. The symmetry $\Theta_{g_1}$, which permutes the $x$ and $y$ axis, maps the vector-field $f_1(x) = A_1 x$ to the vector-field $f_2(x) = A_2 x$ and does not change the vector-field $f_3(x) = A_3 x$. Thus $g_1 \in G$ corresponds to the permutation $\pi_{g_1} = (1 \ 2 \ 3)$. The identity matrix $\Theta_{g_0} = \Omega_{g_0} = I$ corresponds to the identity permutation $\pi_{g_0}$.

The index set $I$ has two orbits $G(1) = \{1, 2\} \text{ and } G(3) = \{3\}$ since the symmetry $g_1$ relates $A_1$ and $A_2$ as $A_2 = \Theta_{g_1} A_1 \Theta_{g_1}^{-1}$ and $A_3$ is only related to itself as $A_3 = \Theta_{g_1} A_3 \Theta_{g_1}^{-1}$. We can select $I/G = \{1, 3\}$ as a set of representative indices from each orbit. The stabilizer subgroups $G_1$ and $G_2$ of $A_1$ and $A_2$ respectively, are trivial $G_1 = G_2 = \{g_0\}$ since only the identity element $\Theta_{g_0} = I$ maps $A_1$ and $A_2$ to themselves. The stabilizer group $G_3$ for $A_3$ is the entire group $G = G_3$ since every symmetry $\Theta_g \forall g \in G = \{g_0, g_1\}$ maps $A_3$ to itself as $A_3 = \Theta_{g_1} A_3 \Theta_{g_1}^{-1} = A_3$.

Next we define symmetry for parameter-dependent controllers and Lyapunov functions. A parameter-dependent controller (2) is symmetric if the state-space transformation $\Theta_y$ and input-space transformation $\Omega_y$ preserve the set $\mathfrak{S}_K = \mathfrak{S}_K y$ the controller polytope $K$ for each $y \in G$.

**Definition 2:** The parameter-dependent control law (2) is symmetric with respect to the group $G = \text{Perm}(A, B)$ if for every extreme-controller gain $K_i \in K$ and $g \in G$

\[
\Theta_y K_i \Theta_y^{-1} = K_{\pi_g(i)}. \tag{9}
\]

By definition 2 the state-space transformation $\Theta_y$ and input-space transformation $\Omega_y$ permute the extreme gains $K_i$ of $K$ in the same way that they permute the extreme state-update matrices $A_i$. Thus
the orbits \( G(i) \) for \( i \in I/G \) partition the extreme controller gains \( K_i \) of a symmetric controller into disjoint equivalence classes. Every pair of extreme controller gains \( K_j \) and \( K_k \) for \( j, k \in G(i) \) are related \( K_j = \Omega_j K_k \Theta_j^{-1} \) by some state-space \( \Theta_j \) and input-space \( \Omega_j \) transformation with \( g \in G \).

Similarly we can define a parameter-dependent Lyapunov function. Recall that since \( \text{Perm}(A,B) \) is finite, there exists a basis such that \( \Theta_g \) and \( \Omega_g \) are orthogonal \( \Theta_g^{-1} = \Theta_g \) and \( \Omega_g^{-1} = \Omega_g^\top \).

**Definition 3:** The parameter-dependent Lyapunov function (3) is symmetric with respect to the group \( G = \text{Perm}(A,B) \) if for every extreme-matrix \( P_i \) and \( g \in G \)

\[
\Theta_g P_i \Theta_g^\top = P_{g \pi_i}(g),
\]

where \( \Theta_g \) are orthogonal.

By definition 3 the state-space transformation \( \Theta_g \) permutes the extreme Lyapunov matrices \( P_i \) in the same way that they permute the extreme state-update matrices \( A_i \). Thus the orbits \( G(i) \) for \( i \in I/G \) partition the extreme Lyapunov matrices \( P_i \) of a symmetric controller into disjoint equivalence classes. Every pair of extreme matrices \( P_i \) and \( P_k \) for \( g \), \( k \in G(i) \) are related by a state-space transformation \( \Theta_g \) for \( g \in G \).

The following theorem shows that symmetric linear parameter varying systems have symmetric parameter-dependent controllers and Lyapunov functions.

**Theorem 1:** The LPV system (1) has a stabilizing parameter-dependent controller (2) with a parameter-dependent Lyapunov function (3) if and only if it has a symmetric controller and Lyapunov function.

**Proof:** If the LPV system (1) has a symmetric stabilizing parameter-dependent controller (2) and corresponding symmetric parameter-dependent Lyapunov function (3) then it obviously has a stabilizing parameter-dependent controller and corresponding Lyapunov function. We constructively prove that the converse also holds.

Suppose the system (1) has a stabilizing parameter-dependent controller (2) certified by a parameter-dependent Lyapunov function (3). From the Lyapunov matrix \( P_i \) we construct the following symmetric Lyapunov matrix

\[
P_i^{-1} = \frac{1}{|G|} \sum_{h \in G} \Theta_h P_{h \pi_i}(h) \Theta_h
\]

and the corresponding controller gains

\[
K_i P_i^{-1} = \frac{1}{|G|} \sum_{h \in G} \Omega_h K_{h \pi_i}(h) \Theta_h
\]

where \( \Theta_h \) and \( \Omega_h \) for \( h \in G \) are symmetries of the LPV system. The Lyapunov matrices (11) are constructed by averaging all the candidate matrices \( P_{h \pi_i}(h) \) with \( h \in G \) for the extreme state-update matrix \( A_i = \Theta_i K_{h \pi_i}(h) \Theta_i \). Likewise the gains \( K_i \) are the average of the candidate controller gains \( K_i = \Omega_i K_{h \pi_i}(h) P_{h \pi_i}(h) \). Therefore \( 1 \) for \( g \in G \).

First we show that the controller \( u(t) = \hat{K}(\xi)x(\xi) \) with \( \hat{K}(\xi) = \sum_{\xi \in \xi} \xi_{\pi_i} \hat{K}_i \) and Lyapunov function \( V(x,\xi) = x^TP(\xi)x \) with \( P(\xi) = \sum_{\xi \in \xi} \xi_{\pi_i} P_i \) satisfy (4) since the matrices \( P_i \) are convex combinations of matrices \( P_i \) and therefore by the spectral properties of Hermitian matrices \( \min_{e \in \xi} \lambda_{\min}(P_i) \geq \min_{e \in \xi} \lambda_{\min}(P_i) \geq \alpha_1 \) and \( \max_{e \in \xi} \lambda_{\max}(P_i) \leq \max_{e \in \xi} \lambda_{\max}(P_i) \leq \alpha_2 \).

Next we show that (5) is satisfied. Note that the following linear matrix inequalities hold

\[
\frac{1}{|G|} \sum_{g \in G} \Phi_g\Pi_{g \pi_i}(g)P_{g \pi_i}(g)^{-1} \geq 0
\]

for all \( i, j \in I \) since \( K_i \) are stabilizing controller gains certified by the Lyapunov matrices \( P_i \). Thus the following matrix inequality holds

\[
\frac{1}{|G|} \sum_{g \in G} \Phi_g\Pi_{g \pi_i}(g)P_{g \pi_i}(g)^{-1} \geq 0
\]

for any \( i, j \in I \). By the Schur complement of this inequality we have \( A_i + B \Omega_i P_i (A_i + B \Omega_i)^\top \geq P_i \), This means that \( K(\xi) \) is a symmetric stabilizing parameter-dependent controller certified by the symmetric Lyapunov function \( V(x,\xi) \).

We conclude that if (1) has a stabilizing parameter-dependent controller (2) with Lyapunov function (3) then it has a symmetric stabilizing parameter-dependent controller and Lyapunov function.

**Theorem 1** shows that symmetric LPV systems (1) have symmetric parameter-dependent controllers (2) and Lyapunov functions (3). Next we establish that the LMI (6) permits symmetric solutions. The following lemma shows that the feasible region of the linear matrix inequalities (6) is symmetric.

**Lemma 1:** Let \( Q \) and \( R \) satisfy \( \Theta_g Q \Theta_g^\top = Q \) and \( \Omega_g R \Omega_g^\top = R \) for all \( g \in G = \text{Perm}(A,B) \), Let \( E_i, G_i, S_i, \) and \( S_j \) be a solution to the linear matrix inequalities (6) for \( (i,j) \in I^2 \). Then \( E_{g \pi_i}(i) = \Theta_g E_i \Theta_g^\top, G_{g \pi_i}(i) = \Theta_g G_i \Theta_g^\top, S_{g \pi_i}(i) = \Theta_g S_i \Theta_g^\top, \) and \( S_{g \pi_j}(j) = \Theta_g S_j \Theta_g^\top \) is a solution to (6) for \( (g \pi_i(i),g \pi_j(j)) \in I^2 \).

**Proof:**

\[
\Phi_g, \Phi_{g \pi_i}(i), \Phi_{g \pi_j}(j)
\]

By Lemma 1 if we have found a solution to the linear matrix inequality (6) for the pair \( (i,j) \in I \), then we have found a solution to (6) for every triple \( (g \pi_i(i),g \pi_j(j)) \) in the orbit \( G(i,j) \subseteq I^2 \). If \( (i,j) \in I^2 \). Additionally if the set of triples \( G_i, E_i, S_i, \) and \( S_{g \pi_j}(j) \) for every \( i \in I \) satisfies the LMI (6) then the set of triples \( \Theta_g E_i \Theta_g^\top, \Theta_g G_i \Theta_g^\top, \Theta_g S_i \Theta_g^\top, \) and \( \Theta_g S_{g \pi_j}(j) \Theta_g^\top \) is also satisfies (6).

The following theorem shows that the linear matrix inequalities (6) permit symmetric parameterizations of the controller gains \( K_i = E_i G_i^{-1} \) and Lyapunov matrices \( P_i = S_i^{-1} \).

**Theorem 2:** Suppose \( Q \) and \( R \) satisfy \( \Theta_g Q \Theta_g^\top = Q \) and \( \Omega_g R \Omega_g^\top = R \). Then the linear matrix inequalities (6) are feasible if and only if they have a symmetric solution

\[
S_{g \pi_i}(i) = \Theta_g S_i \Theta_g^\top, G_{g \pi_i}(i) = \Theta_g G_i \Theta_g^\top
\]

and \( E_{g \pi_i}(i) = \Theta_g E_i \Theta_g^\top \) for all \( g \in G \) and \( i \in I \).

**Proof:** From Lemma 1 using similar argument as Theorem 3.3 in [21].
Proof: Let $S_i, G_i,$ and $E_i$ for $i \in \mathcal{I}$ be a solution to the linear matrix inequalities (6). By Lemma 1 the design variables

$$\hat{S}_i(g) = \Theta g^{-1}S_{sg(i)}\Theta g$$
$$\hat{G}_i(g) = \Theta g^{-1}G_{sg(i)}\Theta g$$
$$\hat{E}_i(g) = \Omega g^{-1}E_{sg(i)}\Theta g$$

for any $g \in \mathcal{G}$ are also a feasible solutions to (6). By convexity of (6), the average solutions $\bar{S}_i = \sum_{g \in \mathcal{G}} \hat{S}_i(g)/|\mathcal{G}|$, $\bar{G}_i = \sum_{g \in \mathcal{G}} \hat{G}_i(g)/|\mathcal{G}|$, and $\bar{E}_i = \sum_{g \in \mathcal{G}} \hat{E}_i(g)/|\mathcal{G}|$ are feasible. Furthermore $\bar{E}_i, \bar{G}_i,$ and $\bar{S}_i$ are symmetric (12).

Remark 1: Theorem 2 is a generalization of the fixed-space theorem in [21]. If $A = \{A_i\}$ is a singleton set then Theorem 2 is equivalent to Theorem 3.3 of [21].

Theorem 2 means that we can restrict our search for design variables $E_i, G_i,$ and $S_i$ that satisfy the LMI (6) to symmetric solutions. In the next section we show how restricting our search to symmetric design variables $E_i, G_i,$ and $S_i$ reduces the computational complexity.

IV. DESIGN OF SYMMETRIC LPV CONTROLLERS

In this section we propose three complementary methods for using symmetry to reduce the complexity of designing controllers for symmetric linear parameter varying systems. The first method reduces the number of design variables. The second method reduces the dimension of the design variables. The third method reduces the number of linear matrix inequalities that the design variables must satisfy.

A. Reduction of the number of design variables

In this section we show how to reduce the number of design variables $E_i, G_i,$ and $S_i$ for $i \in \mathcal{I}$ in the LMI (6).

Using symmetry we only need to define one set of design variables $\tilde{E}_i, \tilde{G}_i,$ and $\tilde{S}_i$ for a representative $i \in \mathcal{I}$ for each orbit $\mathcal{G}(i)$. These design variables must satisfy the linear matrix inequalities

$$\text{LMI}(\Theta g_i, G_i^{-1}, \Omega_j E_j G_j^{-1}, \Theta g_i S_i \Theta g_i^{-1}, \Theta g_i S_j \Theta g_j^{-1}) \succeq 0$$

(14)

for each $i, j \in \mathcal{I}/\mathcal{G}$, $g \in \mathcal{G}/\mathcal{G}_i$, and $h \in \mathcal{G}/\mathcal{G}_j$. After solving the linear matrix inequalities (14) we set

$$E_j = \Theta g_i E_i \Theta g_i^{-1}, G_j = \Theta g_i G_i \Theta g_i^{-1}$$
$$S_j = \Theta g_i S_i \Theta g_i^{-1}$$

(15)

for all $j \notin /\mathcal{G}$ where $g_{si}(i) = j$ for some $g_i$. The parameter-dependent controller (2) for $K_j = E_j G_j^{-1}$ is stabilizing as certified by the Lyapunov function (3) for $P_j = S_j^{-1}$.

The linear matrix inequalities (14) are the original linear matrix inequalities (6) with an additional constraint (15) that the design variables are symmetric (12) for $g \in /\mathcal{G}_i$. According to Lemma 1, restricting the design variables to be symmetric does not alter the feasibility of the linear matrix inequalities. The linear matrix inequalities (14) are feasible if and only if the original linear matrix inequalities (6) are feasible. This design procedure requires solving $|\mathcal{I}|^2$ linear matrix inequalities in $3|\mathcal{I}/\mathcal{G}|$ design variables instead of $3|\mathcal{I}|$ design variables.

The reduction in the number of design variables is the result of enforcing the constraint (15) that the design variables are symmetric for $g \in /\mathcal{G}_i$. Since $E_i, G_j,$ and $S_j$ for $j \notin /\mathcal{G}(i)$ are then dictated by (15) we can eliminate them from the linear matrix inequalities (14). This method of variable reduction works best when the orbits $\mathcal{G}(i)$ are large $|\mathcal{G}(i)| \approx |\mathcal{G}|$. In this case the number of orbits is small $|\mathcal{I}/\mathcal{G}| \approx |\mathcal{I}|/|\mathcal{G}|$ for a large group $|\mathcal{G}| \gg 1$. Thus the number of design variables $E_i, G_i,$ and $S_i$ is much smaller $3|\mathcal{I}/\mathcal{G}| \ll 3|\mathcal{I}|$.

In the next subsection we describe a complexity reduction method that works best when the orbits $\mathcal{G}(i)$ are small $|\mathcal{G}(i)| \ll |\mathcal{G}|$.

The following example demonstrates the reduction in the number of design variables.

Example 2: The LPV system in Example 1 has $|\mathcal{I}| = 3$ extreme state-update matrices. For each extreme state-update matrix $A_i$ we need to design a controller gain $K_i = E_i G_i^{-1}$ and a Lyapunov function matrix $P_i = S_i^{-1}$ for $i = 1, 2, 3$. Thus we have $3|\mathcal{I}| = 9$ design variables.

Since the extreme state-update matrices $A_1$ and $A_2$ are related $A_2 = \Theta g_1 A_1 \Theta g_1^{-1}$ by the symmetry $\Theta g_1$, we can use the controller gain $K_2 = \Omega g_1 K_1 \Omega g_1^{-1}$ and Lyapunov matrices $P_2 = \Theta g_1 P_1 \Theta g_1^{-1}$. Thus we can eliminate the design variables $E_2, G_2,$ and $S_2$. This leaves $3|\mathcal{I}/\mathcal{G}| = 6$ design variables we must find by solving $|\mathcal{I}| = 9$ LMI (14) where the design variables $E_2, G_2,$ and $S_2$ have been replaced by $\Theta g_1 E_1 \Theta g_1^{-1}, \Theta g_1 G_1 \Theta g_1^{-1},$ and $\Theta g_1 S_1 \Theta g_1^{-1}$ respectively.

B. Reduction of the dimension of the design variables

In this section we show how to use symmetry to reduce the dimension of the design variables $E_i, G_i,$ and $S_i$ for $i \in \mathcal{I}$.

This method of complexity reduction is complementary to the method in Section IV-A since it can be applied to the design variables $E_i, G_i,$ and $S_i$ for $i \in \mathcal{I}/\mathcal{G}$ not eliminated.

The stabilizer subgroup $\mathcal{G}_i$ is the symmetry group of the design variables $E_i, G_i,$ and $S_i$ since $\Theta g_i S_i = S_{g_i} \Theta g_i, \Theta g_i G_i = G_{g_i} \Theta g_i,$ and $\Theta g_i E_i = E_{g_i} \Theta g_i$ for all $g_i \in \mathcal{G}_i$ by the definition of $\mathcal{G}_i$. Therefore we can restrict the design variables $E_i, G_i,$ and $S_i$ to the commutator subspaces of the stabilizer subgroup $\mathcal{G}_i$

$$S_i, G_i \in \mathcal{G}_i(\Theta, \Theta) = \{M \in R^{n \times n} : \Theta g_i M = M \Theta g_i, \text{ for } g_i \in \mathcal{G}_i\}$$
$$E_i \in \mathcal{G}_i(\Theta, \Omega) = \{N \in R^{n \times n} : \Omega g_i N = N \Omega g_i, \text{ for } g_i \in \mathcal{G}_i\}$$

According to Lemma 1, restricting the design variables to be symmetric (12) for $g_i \in \mathcal{G}_i$ does not alter the feasibility of the original LMI's i.e. (6) has a solution $G_i, S_i,$ and $E_i$ if and only if it has a solution in the commutators $G_i, S_i, G_i, S_i, E_i \in \mathcal{G}_i(\Theta, \Theta)$.

Recall that the commutators $G_i(\Theta, \Theta) \subseteq R^{n \times n}$ and $G_i(\Theta, \Omega) \subseteq R^{n \times n}$ are vector-spaces of lower dimension than the vectors-spaces $R^{n \times n}$ and $R^{n \times n}$ which nominally contain the design variables $S_i, G_i \in R^{n \times n}$, and $E_i \in R^{n \times n}$. We can reduce the dimension of the design variables $E_i, G_i,$ and $S_i$ by explicitly expressing them in terms of basis matrices $M_{ij} \in R^{n \times n}$ and $N_{ij} \in R^{n \times n}$ for $G_i(\Theta, \Theta)$ and $G_i(\Theta, \Omega)$ respectively

$$S_i = \sum_{j=1}^p s_{ij} M_{ij}, G_i = \sum_{j=1}^p g_{ij} M_{ij}, \text{ and } E_i = \sum_{j=1}^p e_{ij} N_{ij}$$

where $d_{ij}$ and $d_{ij}$ are the dimensions of the vector-spaces $G_i(\Theta, \Theta)$ and $G_i(\Theta, \Omega)$ respectively. The new design variables are the scalar coefficients $s_{ij}, g_{ij}, e_{ij} \in R$ of the basis-matrices $M_{ij}$ and $N_{ij}$. Methods exist for constructing the basis-matrices $M_{ij}$ and $N_{ij}$ when $\Theta g_i$ and $\Omega g_i$ for $g_i \in \mathcal{G}_i$ are permutation matrices [17]. For general symmetries $\Theta g_i$ and $\Omega g_i$, the design variables $E_i, G_i,$ and $S_i$ can be block-diagonalized by “symmetry adapting” the basis matrices $M_{ij}$ and $N_{ij}$ [18], [20].

This method works best when the orbits $\mathcal{G}(i)$ are small $|\mathcal{G}(i)| \ll |\mathcal{G}|$. In this case the symmetry group $\mathcal{G}_i$ of the design variables $E_i, G_i,$ and $S_i$ is large $|\mathcal{G}_i| = |\mathcal{G}/\mathcal{G}_i|$ when the group is large $|\mathcal{G}| \gg 1$. Thus the commutator spaces $G_i(\Theta, \Theta) \subseteq R^{n \times n}$ and $G_i(\Theta, \Omega) \subseteq R^{n \times n}$ are low-dimensional. The reduction of the dimensions of the design variables can be made without changing the feasibility of the original linear matrix inequalities (6).

The following example demonstrates the reduction in the dimension of the design variables.
Example 3: Consider the LPV system in Example 1. Since the stabilizer group \( G_1 \) of \( i = 1 \in I = \{1, 2, 3\} \) is trivial \( |G_1| = 1 \) we cannot use it to reduce the dimension of the design variables \( E_i \), \( G_i \), and \( S_i \). On the other hand, the non-trivial stabilizer \( G_3 = G \) of \( i = 3 \in I \) can be used to reduce the dimensions of the design variables \( E_3 \), \( G_3 \), and \( S_3 \). Without changing the feasibility of the linear matrix inequalities (14), we can restrict \( E_3 \), \( G_3 \), and \( S_3 \) to lie in the commutator subspace \( G_i(\Theta, \Theta) = G_i(\Theta, \Omega) \) of the subgroup \( G_3 = \{g_0, g_1\} \)

\[
G_i(\Theta, \Theta) = \{ M \in \mathbb{R}^{2 \times 2} : [\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}] M = M [\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}] \}
\]

where \( G_i(\Theta, \Theta) = G_i(\Theta, \Omega) \) since \( \Theta_3 = \Omega_3 \) for all \( g \in G = \{g_0, g_1\} \). The commutator subspace \( G_i(\Theta, \Theta) = G_i(\Theta, \Omega) \) is 2 dimensional with basis matrices

\[
M_{31} = [\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}] \quad \text{and} \quad M_{32} = [\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}] .
\]

Therefore we can write the design variables as

\[
E_3 = [\begin{smallmatrix} e_{31} & e_{32} \\ e_{32} & e_{33} \end{smallmatrix}] , \quad G_3 = [\begin{smallmatrix} g_{31} & g_{32} \\ g_{32} & g_{33} \end{smallmatrix}] , \quad S_3 = [\begin{smallmatrix} s_{31} & s_{32} \\ s_{32} & s_{33} \end{smallmatrix}].
\]

Thus we have reduced the dimension of \( E_3 \in \mathbb{R}^{2 \times 2} \) and \( G_3 \in \mathbb{R}^{2 \times 2} \) from 4 to 2 and the dimension of \( S_3 = S_3^T \in \mathbb{R}^{2 \times 2} \) from 3 to 2. \( \square \)

C. Reduction of the Number of Linear Matrix Inequalities

In this section we show how to reduce the number of linear matrix inequalities (6) that the design variables \( E_i \), \( G_i \), and \( S_i \) must satisfy.

The linear matrix inequalities (6) are related by symmetry. For each \( g \in G \) the linear matrix inequalities for the pair of design variables \( (i,j) \) and \( (\pi(i), \pi(j)) \) are symmetrically equivalent

\[
\text{LMI}(G_{\pi(i)} E_{\pi(i)}, S_{\pi(i)} S_{\pi(j)}) = \Phi \text{LMI}(G_i, E_i, S_i) \Phi^T
\]

where \( \Phi = \text{diag}(\Theta_3, \Theta_3, \Theta_3, \Theta_3) \) and the design variables \( G_i, E_i, \) and \( S_i \) are symmetric (12). In other words the linear matrix inequality (6) for the pair \( (\pi(i), \pi(j)) \) is positive definite if and only if the linear matrix inequality (6) for \( (i,j) \) is positive definite. Therefore if the design variables \( E_i \), \( G_i \), and \( S_i \) for \( i \in I \) are symmetric (12) then only need to satisfy the LMI (6) for one pair \( (i,j) \in I^2/G \) from each orbital \( G(i,j) \). By Lemma 1 this does not change the feasibility of the original LMs (6).

We can combine this result with the reduction in the number of design variables presented in Section IV-A. A representative \( (i,j) \in I^2/G \) from each orbital \( G(i,j) \) can be generated by selecting \( i \in I/G \) and \( j \in I/G \) where \( j = \pi_s(k) \) for some \( k \in I/G \) and \( g = S_i \). Thus the reduced design variables \( E_i, G_i, \) and \( S_i \) for \( i \in I/G \) must satisfy the linear matrix inequalities

\[
\begin{bmatrix}
G_i + G_i^T - S_i - (A_i G_i + B E_i)^T G_i^T - Q^{-1} 0 & 0 \\
(A_i G_i + B E_i) \Theta_i S_i \Theta_i & E_i
\end{bmatrix} \geq 0
\]

(17)

for each \( i,j \in I^2/G \) where \( g \in G \) and \( k \in I/G \) such that \( j = \pi_s(k) \). The dimension of the design variables \( E_i, G_i, \) and \( S_i \) for \( i \in I/G \) can be reduced using the method described in Section IV-B. This design process requires solving \( I^2/G \) linear matrix inequalities (17) in \( 3|I/G| \) design variables.

The following examples demonstrate the reduction in the number of linear matrix inequalities.

Example 4: Consider the LPV system in Example 1. First we need to calculate the orbitals \( G(i,j) \) for all pairs \( (i,j) \in I^2 \). There are \( |I^2/G| = 5 \) orbitals \( G(1,1) = \{(1,1), (2,2)\}, G(1,2) = \{(1,2), (2,1)\}, G(1,3) = \{(1,3), (2,3)\}, G(3,1) = \{(3,1), (3,2)\}, \) and \( G(3,3) = \{(3,3)\} \). We can select one representative pair \( (i,j) \) from each orbital \( G(i,j) \) : \( I^2/G = \{(1,1), (1,2), (1,3), (3,1), (3,3)\} \). For each pair \( (i,j) \in I^2/G \) we build a linear matrix inequalities (17) and replace the design variables \( E_3, G_2, \) and \( S_2 \) with \( \Omega_3^{-1} E_1 \Theta_3, \Theta_3^{-1} E_1 \Theta_3, \) and \( \Theta_3^{-1} S_1 \Theta_3 \) respectively. Thus we solve \( |I^2/G| = 5 \) linear matrix inequalities in \( 3|I/G| = 6 \) design variables instead of the original \( |I^2| = 9 \) inequalities in \( 3|I| = 9 \) design variables. \( \square \)

V. APPLICATION: BUILDING CONTROL

In this section we apply the results of this paper to the control of heating, ventilation, and air conditioning (HVAC) of a building. Symmetry is used to decrease the number of design variables and linear matrix inequalities.

Building HVAC control systems are used to regulate the room temperature in buildings. Figure 2 shows a conceptual diagram of an HVAC system. The HVAC system has a centralized component which includes the compressors, condensers, and ventilation ports and subsystems for each room which include electric heaters, air-conditioner evaporators, ventilation dampers, and the thermal dynamics of the room. The dynamics of an N room building can be model by the following state-space system

\[
\begin{bmatrix}
x_N(t+1) \\
x_{N+1}(t+1) \\
\end{bmatrix} =
\begin{bmatrix}
\begin{bmatrix}
A_{11} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_{NN}
\end{bmatrix} & B \\
0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
x_N(t) \\
x_{N+1}(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
B
\end{bmatrix} u(t)
\]

where \( x_N(t) \in \mathbb{R}^n \) and \( x_{N+1}(t) \in \mathbb{R}^{n+1} \) for \( r = 1, \ldots, N \) are the states of the central dynamics and room dynamics respectively and \( u(t) \in \mathbb{R}^n \) and \( u'(t) \in \mathbb{R}^{n+1} \) for \( r = 1, \ldots, N \) are the central and room control inputs respectively.

The state-update matrices \( A^{tr} = A^{tr}(\rho_r) \) for each room depend on a scalar parameter \( \rho_r \in \rho \) which represents room-size. The coupling matrices \( A^{rs} = A^{rs}(\rho_r, \rho_s) \) depend on the relative sizes \( \rho_r \) and \( \rho_s \) of rooms. The rooms can be different sizes \( \rho_r \neq \rho_s \), however we assume that the dynamics are identical \( A^{tr}(\rho_r) = A^{tr}(\rho_s) \) if rooms \( r \) and \( s \) are the same size \( \rho_r = \rho_s = \rho^l \). Additionally we assume the input matrices \( B' \) for each room are identical \( B' = B'' \) for all \( r, s = 1, \ldots, N \). We conservatively assume the range of room sizes \( \rho \) for each room is the same \( \rho \in \rho \) for \( r = 1, \ldots, N \).

The parameter vector \( \rho \in \mathbb{R}^n \) collects the room sizes \( \rho_r \) for \( r = 1, \ldots, N \). The state-update matrix \( A(\rho) \) belongs to a polytope \( A = \{A(\rho) \in \mathbb{R}^{n \times n} : \rho \in \rho \} \) where \( n = n_0 + N n_1 \) is the number of vertices \( A = A(\rho) \) where each rooms is either small \( \rho_r' = \rho \) or large \( \rho_r' = \rho_l \). The actual state-update matrix \( A(\rho) = A(\xi) \in A \) of the system can be written as a
convex combination \( A(\xi) = \sum_{i \in I} \xi_i A(\rho_i) \) of the vertex state-update matrices \( A_i \) where \( I = \{1, \ldots, 2^N\} \), \( \xi \in \Xi \subset \mathbb{R}^{2^N} \), and \( \rho_i \) is a vertex of the hypercube \([\rho, \bar{\rho}]^N\).

We seek a single parameter-dependent controller \( u(t) = K(\xi) x(t) \) to stabilize the temperature of the \( N \) rooms for buildings with different room sizes \( \rho \in [\rho, \bar{\rho}]^N \). Using a parameter-dependent controller, the control gain \( \bar{K}(\xi) \) can be adjusted after installation while maintaining closed-loop stability [19]. Using the design procedure from Section II-B, we would need to define design variables \( E_i, G_i, \) and \( S_i \) for each of the \( |I| = 2^N \) extreme combinations of room size. The design variables would need to satisfy \( |I^2| = 2^{2N} \) linear matrix inequalities (6). This approach is impractical for even moderately sized buildings.

Instead we can use symmetry to achieve an exponential decrease in the number of design variables and number of linear matrix inequalities. The symmetry group \( \text{Perm}(A, B) \) for this problem contains all the matrices of the form

\[
\Theta = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} I_{m_0} & 0 \\ 0 & I_{m_1} \end{bmatrix},
\]

where \( \Pi \in \mathbb{R}^{N \times N} \) is any permutation matrix on \( \mathbb{R}^N \) and \( \otimes \) is the Kronecker product. The group \( G = \text{Perm}(A, B) \) has \( |G| = N! \) elements. This symmetry group means permuting the rooms does not change the dynamics of all possible combinations of room sizes.

We can permute the rooms without changing the \( B \) matrix in (18). The state-update matrix \( A \) may change, however the range of possible state-update matrices \( A \) does not change.

The symmetries (19) organize the extreme state-update matrices \( A_i = A(\rho_i) \in \text{Perm}(A, B) \) into equivalence classes \( G(i) \). Under symmetry, all extreme state-update matrices \( A_i = A(\rho_i) \) corresponding to buildings with \( M \) small rooms \( \rho_{ri} = \rho \) and \( N - M \) large rooms \( \rho_{ri} = \bar{\rho} \) are equivalent. From each orbit \( G(i) \) we can choose the representative extreme state-update matrix \( A_i = A(\rho_i) \) for which the first \( M \) rooms are small and the remaining \( N - M \) rooms are large for \( r = M + 1, \ldots, N \) and \( \rho_{ri} = \bar{\rho} \) for \( r = 1, \ldots, M \). Thus we have the following set of representative indices

\[
I/G = \{ i \in I : \rho_{ri} = \rho \text{ for } r = 1, \ldots, M \text{ and } \rho_{ri} = \bar{\rho} \text{ for } r = M + 1, \ldots, N \}. \tag{20}
\]

The index set \( I \) has \( |I/G| = N + 1 \) orbits of size \( |G(i)| = \binom{M}{N-M} \) for \( M = 0, \ldots, N \).

According to Section IV-A, we only need to define one set of design variables \( E_i, G_i, \) and \( S_i \) for one index \( i \in I/G \) in the representative set \( I/G \). From the definition (20) of \( I/G \) this means we can design the controller gains \( K_i = E_i G_i^{-1} \) and Lyapunov matrices \( P_i = S_i^{-1} \) assuming the rooms are sorted by size. Then for any building containing \( M \) small rooms and \( N - M \) large rooms we obtain the controller gain \( K_j \) and Lyapunov matrix \( P_j \) by permuting the room order \( K_j = \Omega^{-1} K_i \Theta \) and \( P_j = \Theta^T P_i \Theta \) for the controller gain \( K_i \) and Lyapunov matrix \( P_i \) with the sorted room sizes.

We can also use symmetry to reduce the number of linear matrix inequalities (6). According to Section IV-C we only need to solve the linear matrix inequalities (6) for one representative pair \((i, j) \in I^2/G \) from orbital \( G(i) \). Using standard techniques from computational group theory we can find pairs \( i, k \in I/G \) and \( \Theta_k \) for \( g \in G \) satisfies \( \sigma_k(k) = j \) for each \((i, j) \in I^2/G \). It can be shown that \( I^2/G \) has size \( O(|I^2/G|) = N^3 \).

Thus using symmetry we have reduced the number of design variables from \( O(|I|) = 2^N \) to \( O(|I/G|) = N \) and reduced the number of linear matrix inequalities from \( O(|I^2|) = 2^{2N} \) to \( O(|I^2/G|) = N^3 \).

VI. Conclusions

In this paper we modified a linear matrix inequality based procedure for designing parameter-dependent controllers and Lyapunov functions. Our modification exploits problem symmetry to reduce the computational complexity of controller design. We describe how to reduce the number of design variables, the dimension of the design variables, and the number of linear matrix inequalities that the design variables must satisfy. We applied our controller design technique to an HVAC control problem. Our method produced an exponential decrease in computational complexity.

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References