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Solution to the HJB Equation for LQR-Type Problems on Compact Connected Lie Groups ⋆

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Abstract

In this paper, we provide a solution to the HJB equation associated with LQR-type problems for fully-actuated left-invariant control systems on compact connected Lie groups. We obtain the corresponding algebraic/differential Riccati equation and in turn generalize some of the results in [3] to a broader class of Lie groups. Closed-loop stability results are also derived.

Key words: Compact connected Lie groups; HJB equation; LQR-Type problems; Optimal control; Value function.

1 Introduction

The Hamilton-Jacobi-Bellman (HJB) equation on smooth manifolds has received considerable attention (see, e.g., [1, Chapter 17], [16] and references therein), yet the HJB equation on Lie groups has not been as extensively studied. In the literature, the HJB equation on Lie groups has been considered from both the control [3,11,12,14,15] and filtering [13] perspectives. Except for the case of $\mathbb{R}^n$, in general there is no well-known form for the value function for LQR-type problems on Lie groups. Some results in this direction have been obtained for the case of the Lie group SO(3) in [11], which used the Euclidean distance obtained by embedding SO(3) in $\mathbb{R}^{3\times 3}$, and later in [3], which used the geodesic distance.

In this paper, we provide a solution to the HJB equation associated with LQR-type problems for fully-actuated left-invariant control systems on compact connected Lie groups. [1] We generalize the results of [3], which are applicable to SO(3), to a broader class of Lie groups. Furthermore, for the special class of discounted-cost infinite-horizon LQR-type problems, we demonstrate exponential stability with no restrictions on the discount rate. Note that this property fails to hold more generally. Specifically, when there is an inverse of an exponential function multiplying the state and control costs, it is possible to have a mode of the closed-loop system that diverges at an exponential rate slower than half of the discount rate. For more details, we refer the interested reader to [6,10].

We remark that even though our generalization is based on the ideas of [3], our results are not merely a straightforward extension of those in that work. In particular, to obtain these results we need to exploit a much more involved technical machinery. Furthermore, the results obtained in this paper are useful not only from the theoret-

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[1] Examples of these include SO(n) and Sp($2n$).

ical viewpoint but are also practical; apart from SO(3), which arises in attitude control problems, the results are also applicable to $\mathrm{SU}(2^n)$, which arises in quantum control problems [14,15]. In addition, they are also useful in constructing a Lyapunov function for such control systems.

The classes of optimal control problems (OCPs) that we consider in this paper are well-defined under the assumption that the optimal trajectory does not encounter any singularities of the exponential map. This assumption helps in exploiting the inherent geometric nature of the problem in an efficient way. For example, the work of [11,12] considers an LQR-type problem for SO(3) that is well-defined everywhere, by embedding it in a finite-dimensional Euclidean space and then using the metric induced from the Frobenius norm. However, this requires the use of a non-differentiable value function, for which the concept of a viscosity solution associated to the HJB equation has to be used, as in [12]. For further discussion, see [3].

1.1 Notation

Most of the notation is standard, with a few exceptions. We will denote an n-dimensional compact connected Lie group by G and its Lie algebra by $\mathfrak{g}$. The identity element of G is denoted by e. The tangent space of G at $g \in G$ is denoted by $T_g G$. The left and right translation maps on G are denoted by L_g and R_g, respectively. The tangent map (differential) of L_g at $h \in G$ is denoted by $T_h L_g$ and the cotangent map (which is defined to be the adjoint of the tangent map $T_h L_g$) is denoted by $T_h^* L_g$. The differential of a smooth mapping $f : M \to N$ between two smooth manifolds M and N at $x \in M$ is denoted by $\mathrm{d}\, f_x$. Similarly, the differential of a smooth mapping $f : M \times [0,1] \to N$ (where M and N are smooth manifolds) at $(x,t) \in M \times [0,1]$ with respect to its first argument is denoted by $\mathrm{d}_1\, f_{(x,t)}$. The exponential map is denoted by exp and the inverse of the exponential map is denoted by log. The Ad-invariant inner product (positive definite symmetric bilinear form) on $\mathfrak{g}$ is denoted by $\langle \cdot, \cdot \rangle$. The natural pairing between the tangent and cotangent spaces is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. The map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$. For $x \in \mathfrak{g}$, the adjoint action of x on $\mathfrak{g}$ is given by the endomorphism $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$, with $\mathrm{ad}_x(y) = [x, y]$ for all $y \in \mathfrak{g}$, where $[\cdot, \cdot]$ denotes the Lie bracket.

2 Preliminaries

A fully-actuated left-invariant control system on G is given by

$$\dot{g} = T_e L_g(u), \tag{1}$$

where $g(\cdot) \in C^1([0,T], G)$ and u is a curve in $\mathfrak{g}$. More precisely, if $\mathfrak{g} = \mathrm{span}\{e_1, \ldots, e_n\}$, then u is given by

$$u(t) = \sum_{i=1}^{n} u^i(t) e_i,$$

where the n-tuple of control inputs $[u^1 \ \cdots \ u^n]^T$ is an element of $\mathbb{R}^n$.

A compact connected Lie group is a connected Lie group which has a compact Lie algebra. A Lie algebra $\mathfrak{g}$ is compact if there exists an ad-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, i.e., $\langle \cdot, \cdot \rangle$ satisfies the relationship

$$\langle \mathrm{ad}_x(y), z \rangle + \langle y, \mathrm{ad}_x(z) \rangle = 0 \tag{2}$$

for all $x, y, z \in \mathfrak{g}$. The Ad-invariant inner products on a Lie algebra are in bijective correspondence with the bi-invariant metrics on the corresponding Lie group [7, Proposition 18.3]. Furthermore, in the case where the Lie group is connected, as is the case for G, an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ induces a bi-invariant metric on G if and only if the linear map $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is skew-adjoint for all $x \in \mathfrak{g}$, i.e., it satisfies the relationship

$$\langle \mathrm{ad}_x(y), z \rangle = -\langle y, \mathrm{ad}_x(z) \rangle$$

for all $x, y, z \in \mathfrak{g}$ [9, Lemma 7.2], which is equivalent to (2). Therefore, $\langle \cdot, \cdot \rangle$ is Ad-invariant only if it is ad-invariant and the converse holds since G is connected.

Note that if the Lie algebra of a compact Lie group is simple, then the bi-invariant metric is unique up to multiplication by a positive constant [9, Lemma 7.6]. Furthermore, it is also a well-known result in the theory of Lie algebras that on a compact and semisimple Lie algebra, the Killing form is negative definite. [2] Since simple Lie algebras are also semisimple, from our above discussion, it now follows that the only possible choice for an ad-invariant inner product on a compact and simple Lie algebra is given by the negative of the Killing form up to a positive multiple.

We present the following results, which will be useful in proving subsequent results in the paper. They are proven for the general case of any Lie group, and are similar to the results of [5, Theorems 2-3].

Theorem 1 *Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. For $z \in \mathbb{C}$ near zero, consider the function*

$$f_1(z) := \frac{z}{1 - e^{-z}} = \sum_{n=0}^{\infty} \frac{(-1)^n \mathcal{B}_n}{n!} z^k, \tag{3}$$

[2] The Killing form $\kappa(x,y) := \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$ is an ad-invariant (and also an Ad-invariant) symmetric bilinear form on $\mathfrak{g}$.

where $\{\mathcal{B}_n\}_{n=0}^{\infty}$ are the Bernoulli numbers. If $g(\cdot) \in C^1([0,1], G)$ is a trajectory of (1) obtained using u, which never passes through the singularity of the exponential map, then

$$\dot{c} = f_1(\mathrm{ad}_c)(u) = \sum_{n=0}^{\infty} \frac{(-1)^n \mathcal{B}_n}{n!} (\mathrm{ad}_c)^n(u), \qquad (4)$$

where $c(t) := \log(g(t))$.

PROOF. Following the steps of the proof in [8, Lemma 4.27] for any $x \in \mathfrak{g}$ which avoids being a singularity of the exponential map, it can be shown that

$$T_{\exp(x)} L_{\exp(x)^{-1}} \circ \mathrm{d}\exp_x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} (\mathrm{ad}_x)^n = f_2(\mathrm{ad}_x), \qquad (5)$$

where $f_2(z) := f_1(z)^{-1}$. Using (5) and again following the steps in [8, Theorem 4.29], it can now be shown that

$$T_{\exp(c(t))} L_{\exp(c(t))^{-1}} \left(\frac{d}{dt} \exp(c(t)) \right) = f_2(\mathrm{ad}_{c(t)})(\dot{c}(t)),$$

implying that

$$T_g L_{g^{-1}}(\dot{g}) = f_2(\mathrm{ad}_c)(\dot{c}).$$

From (1),

$$u = T_g L_{g^{-1}}(T_e L_g(u)) = f_2(\mathrm{ad}_c)(\dot{c}).$$

Observe that $f_1 \cdot f_2 \equiv 1$. Therefore,

$$\dot{c} = f_1(\mathrm{ad}_c)(u) = \sum_{n=0}^{\infty} \frac{(-1)^n \mathcal{B}_n}{n!} (\mathrm{ad}_c)^n(u). \qquad \square$$

We now define the distance between the identity element $e \in G$ and an element $g \in G$ as follows

$$d(e, g) := \sqrt{\langle \log(g), \log(g) \rangle}, \qquad (6)$$

whenever the inverse of the exponential map log is well-defined. The distance (6) defines an appropriate metric because it is closely related to the notion of geodesic distance. In fact, if $\gamma : [0,1] \to G$ is a minimizing geodesic, with $\gamma(0) = g_1$ and $\gamma(1) = g_2$, then the geodesic or Riemannian distance between the elements $g_1, g_2 \in G$ is given by

$$d_g(g_1, g_2) = \sqrt{\langle \log(g_1^{-1} g_2), \log(g_1^{-1} g_2) \rangle}, \qquad (7)$$

whenever the inverse of the exponential map log is well-defined.

Theorem 2 *If $g(\cdot) \in C^1([0,1], G)$ is a trajectory of (1) obtained using u, which never passes through the singularity of the exponential map, then*

$$\frac{d}{dt} d^2(e, g(t)) = \langle c(t), u(t) \rangle. \qquad (8)$$

PROOF. Note that

$$\frac{d}{dt} \frac{1}{2} d^2(e, g(t)) = \frac{d}{dt} \frac{1}{2} \langle c(t), c(t) \rangle = \langle c(t), \dot{c}(t) \rangle.$$

Using Theorem 1, we have

$$\begin{aligned} \langle c, \dot{c} \rangle &= \left\langle c, \sum_{n=0}^{\infty} \frac{(-1)^n \mathcal{B}_n}{n!} (\mathrm{ad}_c)^n(u) \right\rangle \\ &= \langle c, u \rangle + \left\langle c, \sum_{n=1}^{\infty} \frac{(-1)^n \mathcal{B}_n}{n!} (\mathrm{ad}_c)^n(u) \right\rangle. \end{aligned}$$

The ad-invariance property of $\langle \cdot, \cdot \rangle$ implies that

$$\langle c, \dot{c} \rangle = \langle c, u \rangle.$$

Therefore,

$$\frac{d}{dt} \frac{1}{2} d^2(e, g(t)) = \langle c(t), u(t) \rangle. \qquad \square$$

3 Optimal control problems

In this section, we consider two types of LQR-type OCPs. Both are of the form

$$\min_u J \qquad (9a)$$
$$\text{s.t. } \dot{g}(t) = T_e L_{g(t)}(u(t)), \; g(0) = g_0. \qquad (9b)$$

In the first, we minimize a cost function over a finite time horizon; in the second, we minimize a cost function over an infinite time horizon with an exponential discount rate.

3.1 Finite-horizon LQR-type problem

The cost functional in (9a) is given by

$$\begin{aligned} J = &\frac{k_f}{2} d^2(e, g(1)) \\ &+ \frac{1}{2} \int_0^1 \left[q(t) d^2(e, g(t)) + r(t) \langle u(t), u(t) \rangle \right] dt, \end{aligned} \qquad (10)$$

where $k_f \geq 0$, $q(t) > 0$, and $r(t) > 0$ for all $t \in [0,1]$.

The HJB equation for the OCP (9) is

$$V_t + \min_u H(g, u, \mathrm{d}_1 V_{(g,t)}) = 0, \ V(g, 1) = K(g), \quad (11)$$

where the pre-Hamiltonian and the terminal cost are given, respectively, by

$$\begin{aligned} H(g, u, \mathrm{d}_1 V_{(g,t)}) &= \frac{1}{2}[qd^2(e, g) + r\langle u, u\rangle] \\ &\quad + \langle\langle \mathrm{d}_1 V_{(g,t)}, T_e L_g(u)\rangle\rangle, \quad (12) \\ K(g) &= \frac{k_f}{2} d^2(e, g), \quad (13) \end{aligned}$$

and $V : G \times [0, 1] \to \mathbb{R}$ is the value function. Since we can uniquely write $\mathrm{d}_1 V_{(g,t)} = T_g^* L_{g^{-1}}(\mathrm{d}_1 V_{(e,t)})$, using the invariance of $\langle\langle \cdot, \cdot\rangle\rangle$, we obtain the left-trivialized pre-Hamiltonian,

$$\begin{aligned} H^{l-t}(g, u, \mathrm{d}_1 V_{(e,t)}) &= \frac{1}{2}[qd^2(e, g) + r\langle u, u\rangle] \\ &\quad + \langle\langle \mathrm{d}_1 V_{(e,t)}, u\rangle\rangle. \quad (14) \end{aligned}$$

Using the isomorphism $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*$, determined by $\langle \cdot, \cdot\rangle$, the minimization of (14) with respect to u yields

$$u^* = -\frac{1}{r}\mathbb{I}^{-1}(\mathrm{d}_1 V_{(e,t)}), \quad (15)$$

where u^* denotes the optimal control. We now see that the HJB equation (11) reduces to

$$V_t + H^{l-t,*}(g, \mathrm{d}_1 V_{(e,t)}) = 0, \ V(g, 1) = K(g), \quad (16)$$

where the left-trivialized Hamiltonian is given by

$$\begin{aligned} H^{l-t,*}(g, \mathrm{d}_1 V_{(e,t)}) &= \frac{q}{2} d^2(e, g) \\ &\quad - \frac{1}{2r}\langle\langle \mathrm{d}_1 V_{(e,t)}, \mathbb{I}^{-1}(\mathrm{d}_1 V_{(e,t)})\rangle\rangle. \end{aligned}$$

Theorem 3 *The optimal control for the OCP* (9) *with cost functional* (10)*, is given by*

$$u^*(t) = -\frac{p(t)}{r(t)} \log(g(t)), \quad (17)$$

where $p(\cdot) \in C^1([0, 1], \mathbb{R})$ *is the positive solution to the following differential Riccati equation*

$$\dot{p}(t) - \frac{p(t)^2}{r(t)} + q(t) = 0, \ p(1) = k_f, \quad (18)$$

under the assumption that $g(\cdot) \in C^1([0, 1], G)$ *never passes through the singularity of the exponential map.*

PROOF. Let

$$V(g, t) = \frac{p(t)}{2} d^2(e, g). \quad (19)$$

If $g(\cdot) \in C^1([0, 1], G)$ is a trajectory of (1) obtained using u, which never passes through the singularity of the exponential map, then using Theorem 2,

$$\dot{V} = p\langle \log(g), u\rangle + \frac{\dot{p}}{2} d^2(e, g). \quad (20)$$

It follows from the discussion after (13) and before (14), and using the isomorphism $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*$, that

$$\dot{V} = \left\langle \mathbb{I}^{-1}(\mathrm{d}_1 V_{(e,t)}), u\right\rangle + \frac{\dot{p}}{2} d^2(e, g). \quad (21)$$

Together, (20) and (21) imply that

$$\mathrm{d}_1 V_{(e,t)} = \mathbb{I}(p \log(g)). \quad (22)$$

The result follows from the substitution of (22) into (15) and (16) along with the use of the fact that $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*$ is an isomorphism determined by $\langle \cdot, \cdot\rangle$ and also using the specific form of the value function in (19). □

3.2 Discounted-cost infinite-horizon LQR-type problem

Here we consider the cost functional in (9a) to be given by

$$J = \frac{1}{2}\int_0^\infty e^{-\gamma t} \left[qd^2(e, g(t)) + r\langle u(t), u(t)\rangle\right] dt, \quad (23)$$

where $\gamma \geq 0$ is the discount rate, $q > 0$, and $r > 0$. The HJB equation for the OCP (9), with cost functional (23) reads as follows [2, p. 104], [4, Section 10.1]

$$-\gamma V + \min_u H(g, u, \mathrm{d}_1 V_{(g,t)}) = 0, \quad (24)$$

with the terminal condition

$$\lim_{t\to\infty} V(g(t)) = 0. \quad (25)$$

It can again be shown that the optimal control is given by (15), so (24) reduces to

$$-\gamma V + H^{l-t,*}(g, \mathrm{d}_1 V_{(e,t)}) = 0,$$

where the left-trivialized Hamiltonian is given by

$$\begin{aligned} H^{l-t,*}(g, \mathrm{d}_1 V_{(e,t)}) &= \frac{q}{2} d^2(e, g) \\ &\quad - \frac{1}{2r}\langle\langle \mathrm{d}_1 V_{(e,t)}, \mathbb{I}^{-1}(\mathrm{d}_1 V_{(e,t)})\rangle\rangle. \end{aligned}$$

Theorem 4 *The optimal control for the OCP* (9) *with cost functional* (23)*, is given by*

$$u^*(t) = -\frac{p}{r}\log(g(t)), \tag{26}$$

where p is the positive solution to the following equation

$$-\gamma p - \frac{p^2}{r} + q = 0, \tag{27}$$

under the assumption that $g(\cdot) \in C^1([0,\infty), G)$ *never passes through the singularity of the exponential map.*

PROOF. Let

$$V(g) = \frac{p}{2}d^2(e, g).$$

The remainder of the proof follows the steps of the proof of Theorem 3. □

Theorem 5 *The optimal control* (26) *ensures exponential stability of the equilibrium* $e \in G$*, under the assumption that* $g(\cdot) \in C^1([0,\infty), G)$ *never passes through the singularity of the exponential map.*

PROOF. Let

$$V(g) = \frac{p}{2}d^2(e, g),$$

be the Lyapunov candidate function for the equilibrium $e \in G$. Using Theorem 2, we have

$$\dot{V} = -\frac{p^2}{r}\langle \log(g), \log(g)\rangle = -\frac{2p}{r}V.$$

This proves that V is a Lyapunov function and the desired conclusion follows. □

The following result is specialized for the case where the discount rate is set to zero.

Corollary 6 *Let* $\gamma = 0$*. The optimal control for the OCP* (9) *with cost functional* (23)*, is given by*

$$u^*(t) = -\sqrt{\frac{q}{r}}\log(g(t)), \tag{28}$$

under the assumption that $g(\cdot) \in C^1([0,\infty), G)$ *never passes through the singularity of the exponential map.*

4 Example

Let us consider the case where $G = \mathrm{SO}(3)$. As discussed in Section 2, since $\mathfrak{so}(3)$ is compact and simple and, hence, semisimple, the only possible choice for $\langle \cdot, \cdot \rangle$ is given by a negative multiple of the Killing form. We make the following choice

$$\langle x, y \rangle = -\frac{1}{2}\operatorname{tr}(xy) \tag{29}$$

for x, $y \in \mathfrak{so}(3)$. In this case, the exponential map is the matrix exponential and the inverse of the exponential map is the matrix logarithm. We now see that Propositions 1 and 2 in [3] are special cases of Theorems 3 and 4, respectively.

Note that some terms in the results of Theorems 3 and 4 are off by a scalar multiple as compared to the results in [3]. This is because in [3], the following choice for $\langle \cdot, \cdot \rangle$ is made

$$\langle x, y \rangle = -\operatorname{tr}(xy)$$

for $x, y \in \mathfrak{so}(3)$. The choice of $\langle \cdot, \cdot \rangle$ in (29) ensures that, in addition to being an orthogonal basis, $\{e_i\}_{i=1}^3$ is also an orthonormal basis. Furthermore, we refer the interested reader to [3,5,11,12] for details regarding the singularity of the exponential map.

5 Conclusions

We have provided a solution to the HJB equation associated with LQR-type problems for fully-actuated left-invariant control systems on compact connected Lie groups. The main results obtained in this paper contain the well known algebraic/differential Riccati equation. Finally, we have also generalized some of the results in [3] to a broader class of Lie groups.

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