MITSUBISHI ELECTRIC RESEARCH LABORATORIES
http://www.merl.com

Efficient Estimation and Uncertainty Quantification in Space Mission Dynamics

Grover, P.; Sato, Y.

TR2012-062   August 2012

Abstract

The problem of efficient and accurate orbit estimation of space trajectories is discussed. For highly sensitive low-fuel trajectories designed to exploit the complex nonlinear dynamics of the three-body problem, it is vital to have accurate state estimation during maneuvers and ability to deal with irregular observation update times. For instance, in Halo-orbit insertion and station keeping maneuvers, state estimation errors can propagate quickly. In this paper, we combine an efficient probability propagation method with a homotopy-based posterior computation method. The resulting particle filter is highly accurate even in highly nonlinear regime with intermittent observations, and yet an order of magnitude or more efficient than a generic particle filter implementation.

AIAA/AAS Astrodynamics Specialist Conference

Efficient estimation and uncertainty quantification in space mission dynamics

Piyush Grover*

Mitsubishi Electric Research Labs, Cambridge, MA, USA

Yuki Sato†

Advanced Technology R&D Centre, Mitsubishi Electric Co., Amagasaki City, Japan

The problem of efficient and accurate orbit estimation of space trajectories is discussed. For highly sensitive low-fuel trajectories designed to exploit the complex nonlinear dynamics of the three-body problem , it is vital to have accurate state estimation during maneuvers and ability to deal with irregular observation update times. For instance, in Halo-orbit insertion and station keeping maneuvers, state estimation errors can propagate quickly. In this paper, we combine an efficient probability propagation method with a homotopy-based posterior computation method. The resulting particle filter is highly accurate even in highly nonlinear regime with intermittent observations, and yet an order of magnitude or more efficient than a generic particle filter implementation.

I. Introduction

Nonlinear estimation and uncertainty propagation of orbits has been a topic of much research in the last few decades. There exist a wide variety of techniques for offline and online orbit determination. A filter consists of two stages, *propagation* of the probability distribution to obtain the prior, and *update* of the prior with new information to obtain the posterior probability distribution. Linearized estimation methods based on the Kalman-Filter (such as Extended Kalman Filter, Batch-Kalman Filter) have been used to estimate the state of spacecraft in presence of noise and uncertainty. On the other hand, Monte-Carlo based particle filter approaches have been used to perform very accurate estimation for some mission scenarios, and various design phase studies. For posterior computation, approaches like the Extended Kalman Filter solve a Riccati-type equation, while particle filter methods usually involve a pointwise Bayesian update.

Recently, advances have been made to obtain greater understanding of the planar circular restricted three-body problem (PCR3BP)[1] that has lead to complex mission designs. There

*Research Scientist, Mechatronics group

†Research Engineer, Space Dynamics group

is great interest in utilizing trajectories that can use less fuel than the Hohmann-transfer based trajectories. However, the three-body problem is chaotic and highly sensitive to initial conditions, and various types of noise. The resulting trajectories take less fuel than conventional design, but are by nature designed to be unstable.[2] The conventional methods for estimating trajectories have some important drawbacks. For estimating trajectories that are designed to use the very sensitive dynamics of three-body problem, these methods are either not accurate enough (Kalman Filter based techniques), or have very high computational (Monte-Carlo approaches) demands. Accordingly, there is a need for a method that can efficiently and accurately estimate spacecraft trajectories e.g. from an orbit around the Earth to an orbit around the moon, and can be used on-board for real time estimation or off-board to perform large number of design phase studies.

An EKF-type estimation is based on the premise that the nonlinearity is not too high around the tracked trajectory, and frequent updates are available to correct for errors due to linearized propagation model involved. This assumption can be violated in case of low-fuel mission designs, which exploit inherent nonlinear phenomenon to reduce fuel consumption.

A generic particle filter[3] consists of two stages, propagation of the probability distribution to obtain the prior, and update of the prior with new information to obtain the posterior probability distribution. In a generic particle filter, the former is accomplished by using the method of histograms. We solve the Liouville equation for propagation of probability distribution to obtain the prior. This technique has recently been explored in,[4] and leads to highly efficient prior calculation.

The posterior computation process in a generic particle filter is based on a pointwise implementation of Bayes rule. If the process noise covariance is small. this often leads to a situation where almost all the weight (or probability) is assigned to 1 particle, while most of the particles have negligible weight.[5] For computation of the posterior, we employ the method of homotopy based particle flow. This deterministic method of moving particles upon observation eliminates the problem of particle degeneracy, and sample impoverishment, leading to significant gain in efficiency. This method has recently been developed in,[6] and has also been combined with existing techniques.[7] This method involves finding a vector field F that is used to drive the particles at the time of observation.

We use the well known planar restricted three body problem (PCR3BP) models to demonstrate the efficacy of these methods. We demonstrate the combination of Liouville equation based prior compuation, and homotopy based prior computation yields an estimation process that is significantly efficient than a generic particle filter.

II. Uncertainty Propagation In Nonlinear Systems

Consider a dynamical system with initial and parametric uncertainty with discrete measurements, given by the following equation,

$$\dot{X} = f(X) \tag{1}$$

$$Y_{k+1} = h(X_{k+1}) + \eta_{k+1} \tag{2}$$

where $X \in R^n$, and η is noise sampled from a given distribution, t_k is the time at kth time step.

We are interested in the evolution of an initial probability distribution Ω_0, which gives an initial probability to every point in the phase space. In practical cases, this probability is usually a Gaussian distribution at the beginning of the orbit determination process. If the system is nonlinear, the future iterates are no longer Gaussian, and could be arbitrarily shaped.

In the absence of process noise, the probability distribution follows the Liouville equation. This equation describes the evolution of a phase space and time dependent function $P(x,t)$. The function $P(x,t)$ can be thought of as probability density for our purposes, implying that integral over phase space at anytime equals unity.

$$\frac{\partial P(X,t)}{\partial t} = -\sum_{i=1}^{n} \frac{\partial P(X,t) f_i(X)}{\partial x_i} \tag{3}$$

$$\frac{dP(X,t)}{dt} = -P(X,t)\sum_{i=1}^{n} \frac{\partial f_i(X)}{\partial x_i} = -P(X,t)Tr(D_X f(X)) \tag{4}$$

The equation 4 can be integrated along the system trajectories that solve equation 1.

For autonomous Hamiltonian systems, the r.h.s of equation 4 evaluates to zero, since any time independent Hamiltonian system is divergence-less. This property follows from the fact that phase volume is conserved in such systems. It can be shown as follows:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix} \tag{5}$$

$$f = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix} \tag{6}$$

$$D_X(f) = \begin{pmatrix} \frac{\partial^2 H}{\partial q \partial p} & \frac{\partial^2 H}{\partial p \partial p} \\ -\frac{\partial^2 H}{\partial q \partial q} & -\frac{\partial^2 H}{\partial p \partial q} \end{pmatrix} \tag{7}$$

where $X = (q,p)$, and $H(X)$ is the Hamiltonian of the system. Consider the figure 1 showing the propagation of initial distribution P_0 in time. This is accomplished by exploiting the Liouville equation described above. At initial time $t = t_0$, N points are sampled from P_0. The probabilities of various points are denoted as $p^{k,j}$, where j is the index of points, and k is the index of time steps.

To obtain the probability distribution at a later time $t = t_k$, the points are integrated using the equation of motion given by equation 1, along with the Liouville equation 4. Hence the value of probability distribution is exactly solved along the trajectories. To recover the probability at other points in the phase space at time t_k, a fast nearest-neighbors interpolation algorithm is used. This method of solving for propagated probability distributions has been investigated in,[4] and shown promise in reducing the computation load for the probability propagation step. This method can be compared to traditional Monte-Carlo based prior computation process, in which only the system equations are solved. A histogram based method is then used to update the probabilities at the new time step.

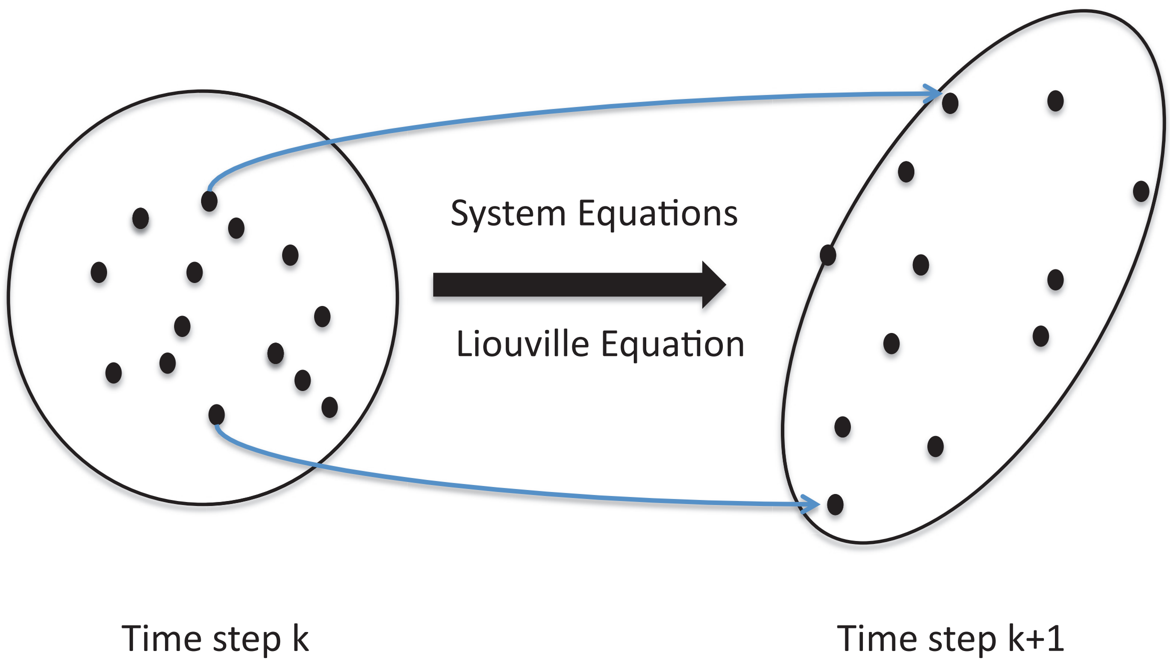

Figure 1. The particles are moved from one time step to another by integrating the system of equations 1. Simultaneously, the Liouville equation 4 is solved to get new probabilities at the updated locations.

III. Posterior computation via homotopy

Here we are concerned with the devising a probability update step which can minimize computational expense, while retaining the high accuracy of the traditional particle filter based approaches. Particle collapse occurs when most of the particles being propagated have probabilities close to zero, and hence, propagating them wastes computational resources, since they are deemed to be unimportant for obtaining the state estimate. In systems such as CR3BP, since the nearby trajectories typically diverge quickly due to sensitive nature, the particle collapse is a particularly spectacular. Following,[6] we use a homotopy based approach to accomplish this. In the homotopy based approach, we solve a a type of differential equation, which moves the particles to a new location once the measurement is received. This motion is done instantaneously in time. Deterministically moving the particles to perform the Bayes rule updation has been recently shown to avoid particle collapse in highly nonlinear systems.[6]

We denote the pseudo-time homotopy variable λ, and use the following representation for unnormalized probability,

$$p(x, \lambda) = g(x)h(x)^{\lambda} \tag{8}$$

where λ goes from 0 to 1, $g(x)$ represents the prior at step $k+1$, and $h(x)$ represents the likelihood function given an observation z_{k+1}. Hence, the homotopy process transforms the prior probability to posterior probability, since the latter is given by the product of prior and likelihood function, as described by Bayes rule. As derived in,[6] an unknown function

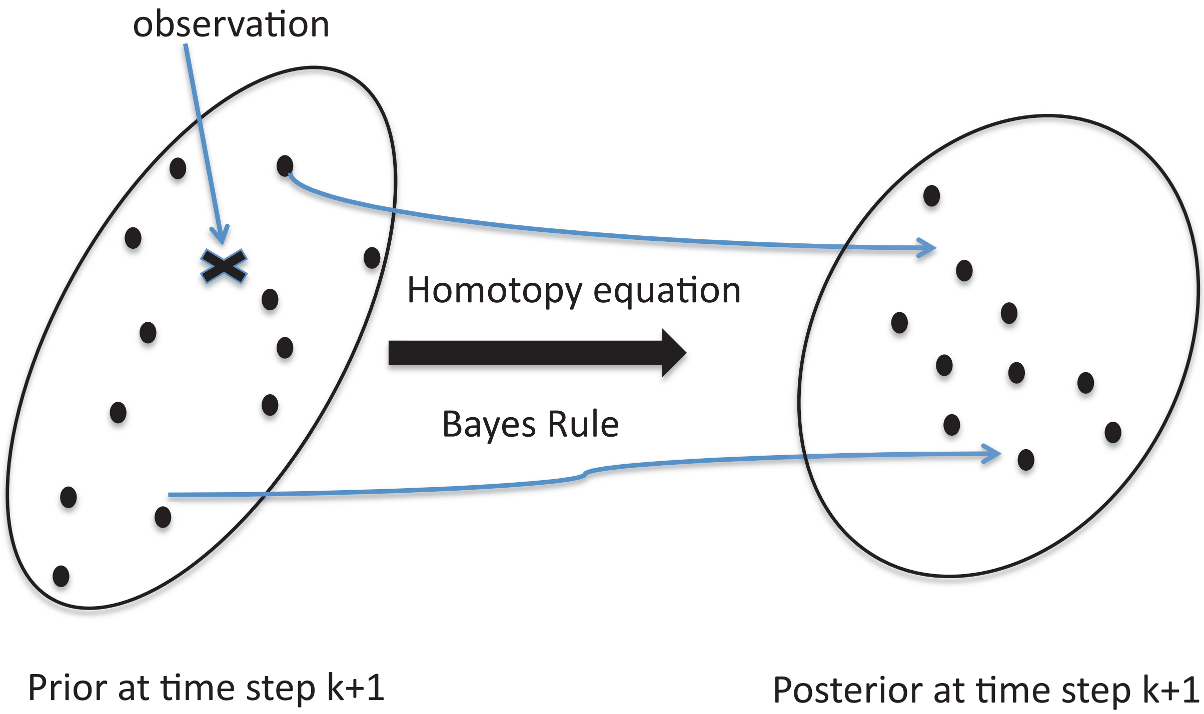

Figure 2. The particles are moved from one time step to another by integrating the system of equations 1.

F is considered as the vector field that will drive the prior particles to posterior particles. This imples we can re-use the Liouville equation equation 4 to represent the probability propagation during the process of homotopy. The original time derivative is replaced by pseudo-time derivative w.r.t λ, and the r.h.s is now the yet unknown function F. Hence we obtain

$$\frac{dx(\lambda)}{d\lambda} = F(x, \lambda) \tag{9}$$

Taking the log on both sides of equation 8, and using equation 9, the following relation is obtained.

$$log(h) + \frac{\partial log(p)}{\partial x} F = -Tr(D_X F) \tag{10}$$

An exact solution exists for the case with linear prior,[8] given by

$$\frac{dx}{d\lambda} = A(\lambda)x + b(\lambda) \tag{11}$$

$$A = (-1/2)PH^T(\lambda HPH^T + R)^{-1}H \tag{12}$$

$$b = (I + 2\lambda A)((I + \lambda A)PH^T R^{-1} z + A\bar{X}) \tag{13}$$

where R is the covariance of the Gaussian noise, P is the error covariance, H is the linear observation funciton and $\bar{X}$ is the state estimate. For the nonlinear case that is of interest to us, the above set of equations can be solved recursively. An EKF is used to obtain the covariance matrix P and updated estimate of $\bar{X}$ and H.

Conceptually, the homotopy based particle flow is a way of representing the 'true' posterior probability in an optimal way. Recently, another approach that uses similar ideas but is based on minimizing a metric has been reported.[7]

IV. The Planar Circular Restricted Three-Body Problem (PCR3BP)

The motion of a *massless* object, P, in presence of two bodies, with masses m_1 and m_2, revolving around each other in a plan can be described in the rotating frame as follows,

$$\ddot{x} - 2\dot{y} = -\bar{U}_x \tag{14a}$$

$$\ddot{y} + 2\dot{x} = -\bar{U}_y \tag{14b}$$

$$\bar{U}(x, y) = -\frac{1}{2}(x^2 + y^2) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} - \frac{1}{2}\mu_1\mu_2 \tag{14c}$$

and r_1 is the distance of P from m_1 and respectively r_2 is the distance of P from m_2. Also, μ_1 and μ_2 are mass parameters. See figure 3. This problem, of describing the motion of the object, P, by Equation 14 is called the planar circular restricted three-body problem (PCR3BP) which will be used to describe the motion of a spacecraft. For a detailed description, see Reference.[1]

A. Insertion into moon orbit

We perform estimation using the methodology described in this paper on a part of a low-fuel GTO to moon trajectory.[9] This section of the trajectory is calculated to minimize the the total fuel required to leave a specified Lyapunov orbit around the L1 point and insert into a 100KM by 800KM elliptical orbit around the moon. The relevant section of the orbit is shown in figure 4

V. Results

We begin by comparing the two methods of obtaining prior probability distribution. For this purpose, we run two versions particle filter. The first one is a generic particle filter using histogram based prior computation, while the second one uses Liouville equation for generating prior probabilities. Since the comparison is intended to highlight the difference between above two approaches, we do not perform any re-sampling, and hence can only the simulation till the number of effective particles drops to 1. Figure 5 shows the estimated trajectories, both of which were started on L1 orbit with an initial and measurement covariance of diag($[1e - 14, 1e - 14, 1e - 10, 1e - 10]$), with update period of about 5 hours.

For comparison of the two versions of the posterior computation process, we investigate the station keeping dynamics around L1 of the PCR3BP. This orbit has been used frequently as a test for nonlinear orbit estimation algorithms, due to presence of instability and moderate nonlinearity. A metric used to quantify the 'usefulness' of particles being used in a particle

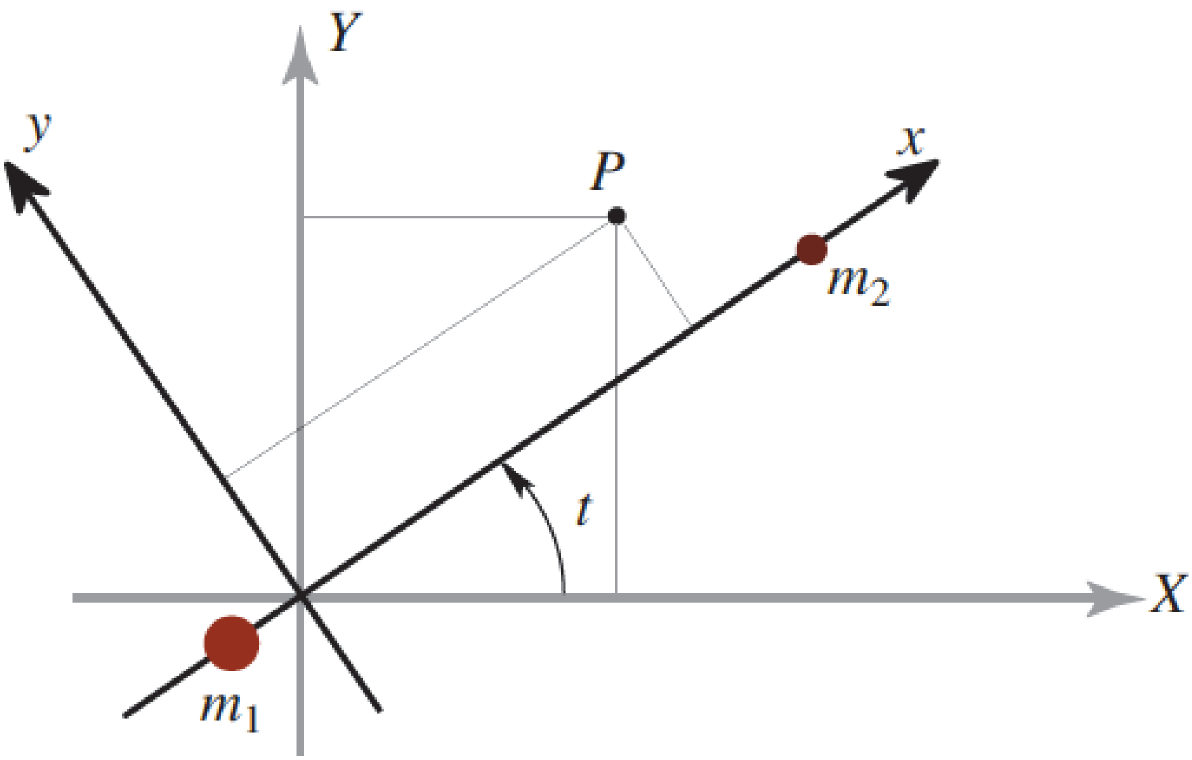

Figure 3. The planar restricted three-body problem 1.

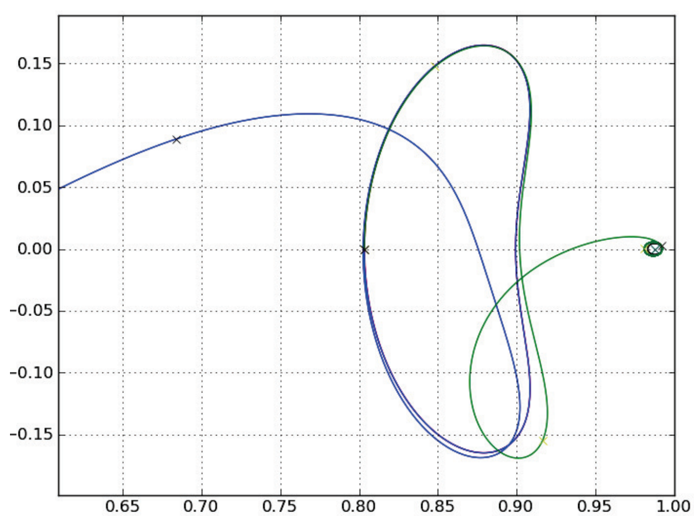

Figure 4. Insertion into (blue) and out of (green) a Lyapunov orbit in the Planar Circular Restricted Three-Body Problem.

filter is the effective number of particles, or its complement, the particle loss. The effective

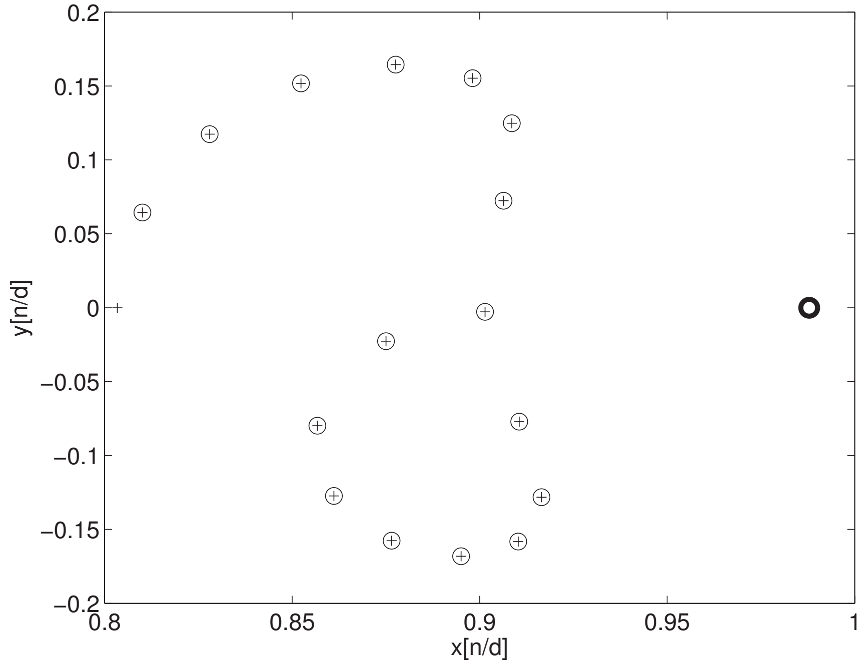

Figure 5. Estimated trajectory using two prior computation techniques, i.e. histogram based (x) filter with 30000 particles and Liouville equation based (o) filter with 6000 particles.

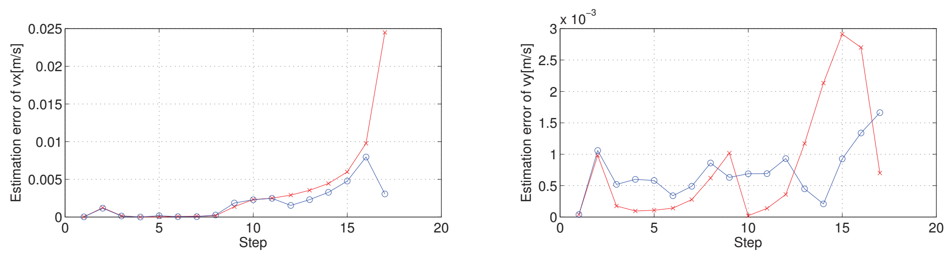

Figure 6. Estimation errors using two prior computation techniques, i.e. histogram based (x) filter with 30000 particles and Liouville equation based (o) filter with 6000 particles.

number of particles at time step k is approximately given by

$$N_{eff,k} = \frac{1}{\sum_{j=1}^{N}(p^{k,j})^2} \tag{15}$$

Figure 7 shows the percentage particle in a station-keeping maneuver around the Lyapunov orbit shown in figure 4. It is clear that the homotopy based method moves the particles to the 'right' places, such that this posterior representation is effective in representing the real posterior probability.

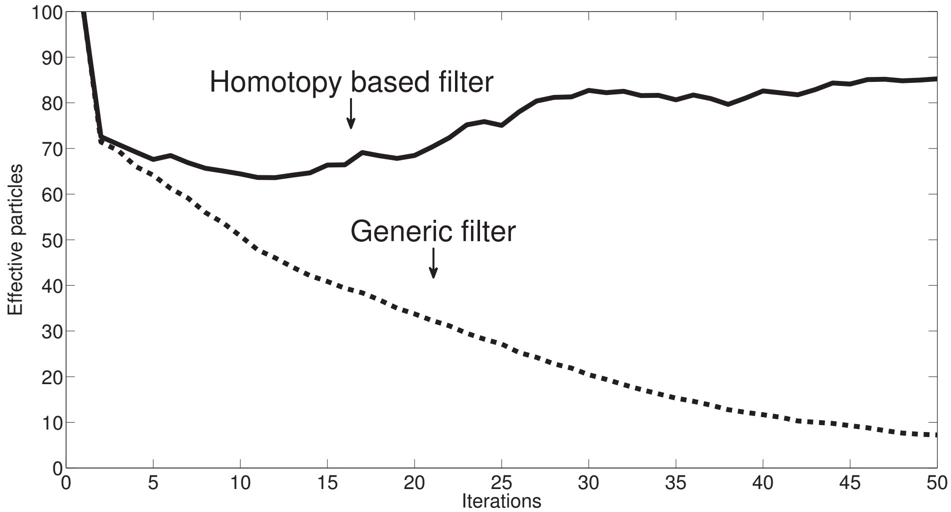

Figure 7. Effective number of particles (percent) using two posterior computation techniques, i.e. generic pointwise update and homotopy equation based update.

VI. Conclusions and Future work

We have combined an efficient uncertainty propagation scheme with a homotopy based probability update method for estimation of an L1 Lyapunov orbit to moon maneuver. We solve the Liouville equation for propagation of probability distribution to obtain the prior. This leads to highly efficient prior calculation, and is shown to reduce the number of particles required significantly. For computation of the posterior, we employ the method of homotopy based particle flow. This method is shown to reduce the particle loss by a significant percentage.

In future work, further analysis of this estimation framework will be performed for the three-dimensional trajectory case, where the efficiency gains are expected to be even higher due to increase in the dimension of state space. Also, since so far we have only compared the number of particles needed for given accuracy, a careful analysis of computation time is also needed.

References

[1] W. Koon, M. Lo, J. Marsden, and S. Ross, *Dynamical Systems, the Three-Body Problem and Space Mission Design*. Marsden Books, 2008.

[2] E. Belbruno, *Capture Dynamics and Chaotic Motions in Celestial Mechanics: With Applications to the Construction of Low Energy Transfers*. Princeton University Press, 2004.

[3] M. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, "A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking," *IEEE Transactions on signal processing*, 2002.

[4] P. Dutta and R. Bhattacharya, "Hypersonic State Estimation using the Frobenius-Perron Operator," *Journal of Guidance, Control and Dynamics*, 2011.

[5] B. Ristic, S. Arunampalam, and N. Gordon, *Beyond the Kalman filter*. Artech House, 2004.

[6] F. Daum, J. Huang, and A. Noushin, "Exact particle flow for nonlinear filters," *Proceedings of the SPIE*, Vol. 7697(04), 2010.

[7] J. Hagmar, M. Jirstrand, L. Svensson, and M. Morelande, "Optimal parameterization of posterior densities using homotopy," *Proceedings of the 14th International Conference on Information Fusion*, 2011.

[8] S. Choi, P. Willet, F. Daum, and J. Huang, "Discussion and Appliciation of the Homotopy Filter," *Proceedings of the SPIE*, Vol. 8050, 2011.

[9] P. Grover and C. Andersson, "Optimized three body assists and manifold transfers in end-to-end lunar mission design," *Proceedings of the 22nd AAS/AIAA Space Flight Mechanics Meeting*, 2012.