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Fast Construction of Covariance Matrices for Arbitrary Size Image Windows

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Abstract

We present a novel, integral image based algorithm to compute feature covariance matrices within all arbitrary size rectangular regions in an image. This technique significantly improves the computational load of covariance matrix extraction process by taking advantage of the spatial arrangement of points. Covariance is an essential measure of how much the deviation of two or more variables or processes match. In our case, these variables correspond to point features such as coordinate, color, gradient, orientation, and filter responses. Integral images are intermediate image representations used for calculation of region sums. Each point of the integral image is a summation of all the points inside the rectangle bounded by the upper left corner of the image and the point of interest. Using this representation, any rectangular region sum can be computed in constant time. We follow a similar idea for fast calculation of region covariance. We construct integral images for all separate features as well as integral images of the multiplication of any two feature combinations. Using these set of integral images and region corner point coordinates, we directly extract the covariance matrix coefficients. We show that the proposed integral image based method decreases the computational load to quadratic time.

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FAST CONSTRUCTION OF COVARIANCE MATRICES FOR ARBITRARY SIZE IMAGE WINDOWS

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ABSTRACT

We present a novel, integral image based algorithm to compute feature covariance matrices within all arbitrary size rectangular regions in an image. This technique significantly improves the computational load of covariance matrix extraction process by taking advantage of the spatial arrangement of points. Covariance is an essential measure of how much the deviation of two or more variables or processes match. In our case, these variables correspond to point features such as coordinate, color, gradient, orientation, and filter responses. Integral images are intermediate image representations used for calculation of region sums. Each point of the integral image is a summation of all the points inside the rectangle bounded by the upper left corner of the image and the point of interest. Using this representation, any rectangular region sum can be computed in constant time. We follow a similar idea for fast calculation of region covariance. We construct integral images for all separate features as well as integral images of the multiplication of any two feature combinations. Using these set of integral images and region corner point coordinates, we directly extract the covariance matrix coefficients. We show that the proposed integral image based method decreases the computational load to quadratic time.

1. INTRODUCTION

Covariance is a statistical measure of the extent to which two random features vary together. Covariance can be a negative, positive or zero number, depending on what is the relation between two feature [**?**]. If the features increase together, the covariance is positive, one feature increases and other decreases, the covariance is negative, and if the two features are independent, the covariance is zero. This is illustrated in Fig. **??**.

There are several advantages of using the covariance matrices as region descriptors. A single covariance matrix extracted from a region is usually enough to match the region in different views and poses. It is possible to discriminate a distribution from other distributions when they do not vary only with their mean.

The covariance matrix provides a natural way of fusing multiple features which might be correlated. Its diagonal entries represent the variance of each feature and the nondiagonal entries represent the scaled correlations. The noise corrupting individual samples are largely filtered out due to the statistical averaging during covariance computation, thus, the covariance matrix values are less sensitive against such uniformly distributed noise.

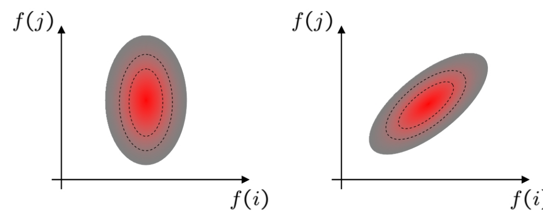

Fig. 1. Distribution of multivariate normal densities. **Left:** Covariance matrix for two features $f(i)$ and $f(j)$ is diagonal, which implies that features do not covary and the off-diagonal elements of the matrix are zero, but feature $f(j)$ varies more than feature $f(i)$. **Right:** Covariance matrix is not diagonal. Instead, $f(i)$ and $f(i)$ have the same variance; $f(i)$ and $f(j)$ tend to increase together. As a result, covariance matrix have identical diagonal elements, and the off-diagonal element would be positive.

In spite of its distinct advantages, computation of the covariance matrices for all possible rectangular regions within a given image is computationally prohibitive using the conventional approaches where the covariance matrix coefficients are extracted using the brute force computation of the mean values and covariances of the features separately for each region as given in the Alg. **??**. Several applications such as detection and segmentation require computation and comparison of covariance matrices of regions. However, conventional approach disregards the fact that there exists a high number of overlaps between those regions and the statistical moments extracted for such overlapping areas can be used to improve the computational load.

Here, we propose a computationally superior method of extracting the covariance matrices of all possible rectangular image regions. This method is based on integral image formulation.

In the next sections, we introduce the region covariance and discuss the integral image based covariance matrix extraction formulation in detail. Then, we give a computational complexity analysis by considering different scenarios.

Algorithm 1.1: CONVENTIONAL($Features$)

```
for each possible image point
do
  for each feature i
  do
    for each window point
      do {Accumulate mean
    Normalize mean
  for each feature i
  do
    for each window point
      do {Compute mean distance
  for each feature i = 1 to d
  do
    for each feature j = i to d
    do
      for each window point
      do
        do {Multiply mean distances
           {Accumulate covariance
      Normalize covariance
```

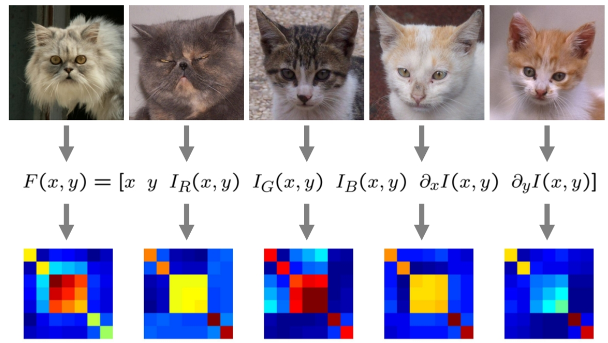

Fig. 2. Corresponding covariance matrices for sample images.

2. COVARIANCE AS A REGION DESCRIPTOR

Let F be the $W \times H \times d$ dimensional feature array extracted from an $W \times H$ image I

$$F(x, y) \models \phi(I, x, y) \tag{1}$$

where the function ϕ assigns array indices x, y to d-dimensional feature vectors that are constructed by any combination of point-wise modalities such as intensity, color, infra-red, multi-spectral values as well as features such as gradients, filter responses, texture scores, etc. For a given rectangular region $R \subset F$, let $\{\mathbf{f}_k\}_{k=1..n}$ be the d-dimensional feature points inside R, and n is the number of points within that region. We represent the region R with the $d \times d$ covariance matrix of the feature points

$$\mathbf{C}_R = \frac{1}{n-1} \sum_{k=1}^{n} (\mathbf{f}_k - \boldsymbol{\mu})(\mathbf{f}_k - \boldsymbol{\mu})^T \tag{2}$$

where $\boldsymbol{\mu}$ is the mean of the points. As shown, the diagonal elements represent the variance for different features we measure. For example, the i^{th} diagonal element represents the variance for the i^{th} feature we measure. The off-diagonal elements represent the covariance between two different features. Covariance matrices for sample images using a set of 7 features are depicted in Fig **??**. Covariance matrix has several important properties:

- If feature i and feature j tend to increase together, then $C_R(i, j) > 0$
- If feature i tends to decrease when feature j increases, then $C_R(i, j) < 0$
- If feature i and feature j are independent, $C_R(i, j) = 0$
- $|C_R(i, j)| \leq \sigma(i)\sigma(j)$, where σ is the standard deviation

The covariance matrices are low-dimensional compared to other region descriptors. Due to symmetry it has only $(d^2 + d)/2$ different values. Whereas if we represent the same region with raw values we need $n \times d$ dimensions, and if we use joint feature histograms we need b^d dimensions, where b is the number of histogram bins used for each feature.

Given a region R, its covariance $\mathbf{C}_R$ does not have any information regarding the ordering and the number of points. This implies a certain scale and rotation invariance over the regions in different images. Nevertheless, if information regarding the orientation of the points are represented, such as the norm of gradient with respect to x and y, the covariance descriptor is no longer rotationally invariant. The same argument is also correct for scale and illumination. Rotation and illumination dependent statistics are important for recognition/classification purposes.

3. INTEGRAL IMAGES FOR FAST COVARIANCE CALCULATION

It is possible to calculate the sum of the values within rectangular windows in linear time without repeating the summation operator for each possible window as shown by Crow [**?**], Veksler [**?**], and Porikli [**?**]. A constant number of operation for each rectangular sum is needed to compute such sums over distinct rectangles many times. A cumulative image function is defined such that each element of this function holds the sum of all values to the left and above of the pixel including the value of the pixel itself. The cumulative image can be computed for all pixels with four arithmetic operations per pixel. We start in the top left corner, keep going first to the right and then down, and use the formula that the value of

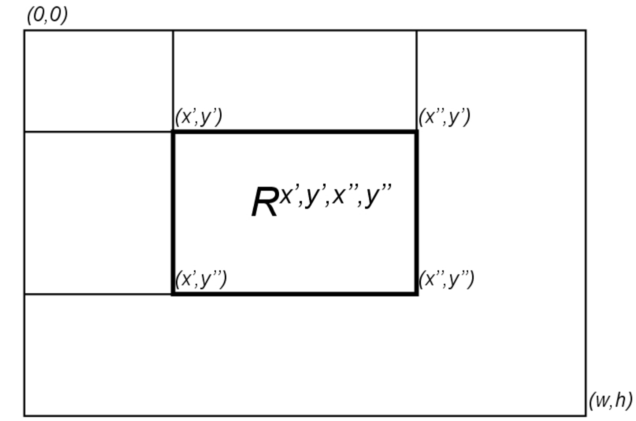

Fig. 3. Integral Image. The rectangle $R(x', y'; x'', y'')$ is defined by its upper left (x', y') and lower right (x'', y'') corners in the image and each point is a d dimensional vector for q .

the cumulative image at the current pixel equal to the addition of the left and the up pixel and subtraction of the upper left pixel's cumulative values. After the cumulative image is computed, the sum of image function in a rectangle can be computed with another four arithmetic operations with appropriate modifications at the border. Thus with a linear amount of computation, the sum of image function over any rectangle can be computed in linear time.

Integral images are intermediate image representations. Later, this idea was extended to fast calculation of region histograms Porikli [**?**]. Here we follow a similar idea for fast calculation of region covariances. Still, the idea presented here is more general than the image sums, which were already published before, and with a series of integral images the covariances are obtained by a few arithmetic operations.

We can rewrite the (i, j)-th element of the covariance matrix defined in (**??**) as

$$C_R(i,j) = \frac{1}{n-1}\sum_{k=1}^{n}(f_k(i)-\mu(i))(f_k(j)-\mu(j)). \quad (3)$$

Expanding the mean and rearranging the terms we can write

$$C_R(i,j) = \frac{1}{n-1}\left[\sum_{k=1}^{n} f_k(i)f_k(j) - \frac{1}{n}\sum_{k=1}^{n} f_k(i)\sum_{k=1}^{n} f_k(j)\right].$$

To find the covariance in a given rectangular region R we have to compute the sum of each feature dimension $f(i)_{i=1..n}$ as well as the sum of the multiplication of any two feature dimensions $f(i)f(j)_{i,j=1..n}$. It is possible to compute these sums with a few arithmetic operations using a series of integral images.

We construct integral images for each feature dimension $f(i)$ and multiplication of any two feature dimensions $f(i)f(j)$. As a result we construct $d + d^2$ integral images. Let P be the $W \times H \times d$ tensor of the integral images along each feature dimension, i.e.,

$$P(x', y', i) = \sum_{x<x', y<y'} F(x, y, i) \quad (4)$$

and Q be the $W \times H \times d \times d$ tensor of the second order integral images, i.e.,

$$Q(x', y', i, j) = \sum_{x<x', y<y'} F(x, y, i)F(x, y, j). \quad (5)$$

In [**?**], it is shown that integral images can be computed in one pass over the image. In our notation $\mathbf{p}_{x,y}$ is the d dimensional vector and $\mathbf{Q}_{x,y}$ is the $d \times d$ dimensional matrix

$$\begin{aligned}\mathbf{p}_{x,y} &= [P(x,y,1)\ldots P(x,y,d)]^T \\ \mathbf{Q}_{x,y} &= \begin{pmatrix} Q(x,y,1,1) & \ldots & Q(x,y,1,d) \\ & \vdots & \\ Q(x,y,d,1) & \ldots & Q(x,y,d,d)\end{pmatrix}. \quad (6)\end{aligned}$$

Note that $\mathbf{Q}_{x,y}$ is a symmetric matrix and $d + (d^2 + d)/2$ passes over the image are enough to compute both P and Q. The computational complexity of constructing the integral images is $O(d^2WH)$.

Let $R(x', y'; x'', y'')$ be the rectangular region, where (x', y') is the upper left coordinate and (x'', y'') is the lower right coordinate, as shown in Figure **??**. The covariance of the region bounded by $(1, 1)$ and (x'', y'') is

$$\mathbf{C}_{R(1,1;x'',y'')} = \frac{1}{n-1}\left[\mathbf{Q}_{x'',y''} - \frac{1}{n}\mathbf{p}_{x'',y''}\mathbf{p}^T_{x'',y''}\right] \quad (7)$$

where $n = x'' \times y''$. Similarly, after a few manipulations, the covariance of the region $R(x', y'; x'', y'')$ can be computed as

$$\begin{aligned}\mathbf{C}_{R(x',y';x'',y'')} = \frac{1}{n-1}\Big[&\mathbf{Q}_{x'',y''} + \mathbf{Q}_{x',y'} \quad (8)\\ &-\mathbf{Q}_{x'',y'} - \mathbf{Q}_{x',y''} \\ &-\frac{1}{n}\big(\mathbf{p}_{x'',y''} + \mathbf{p}_{x',y'} - \mathbf{p}_{x',y''} - \mathbf{p}_{x'',y'}\big) \\ &\cdot\big(\mathbf{p}_{x'',y''} + \mathbf{p}_{x',y'} - \mathbf{p}_{x',y''} - \mathbf{p}_{x'',y'}\big)^T\Big]\end{aligned}$$

where $n = (x''-x') \times (y''-y')$. Therefore, after constructing integral images the covariance of any rectangular region can be computed in $O(d^2)$ time. We give the integral image based covariance computation in Alg. **??**.

Algorithm 3.1: INTEGRAL($Features$)

for each *possible image point*
do
- **for each** *feature* $i = 1$ **to** d
  - **do** {*Accumulate integral* $f(i)$
- **for each** *feature* $i = 1$ **to** d
  - **do**
    - **for each** *feature* $j = i$ **to** d
      - **do** {*Accumulate integral* $f(i)f(j)$

for each *possible image point*
do
- **for each** *feature* i
  - **do**
    - *Get integral values at window corners*
    - *Compute covariance*

4. DISCUSSION

To make an accurate analysis of the computational improvement, we implemented both the conventional and the integral image based methods in Matlab. Fig. **??** shows the CPU times for the extraction of all regions covariance matrices for different image sizes and different region sizes using a fixed (7) number of features. As visible, the proposed method accelerates the extraction process almost 50-to-500 times depending on the image and region size. For instance, the proposed method can extract all the covariance matrices in only 15 seconds for 49×49 target regions. On the other hand, the conventional approach requires around 900 seconds (15 minutes!) for the same task. The reason the proposed method becomes slightly faster for the larger region sizes is that the number of possible regions in the image decreases with the increasing region size.

We also analyzed the effect of the number of features on the computational improvement. The results are given in Fig. **??**. We observed that the proposed method quadratically faster than the conventional method with the increasing number of features.

In summary, we presented a novel and computationally very fast method to compute the covariance matrices of all possible regions in an image. Our intensive simulations prove that the integral image based method can expedite the search process more than hundreds of times in comparison to the existing conventional approaches. In addition, it enables construction of advanced covariance features for further feature detection and classification purposes.

5. REFERENCES

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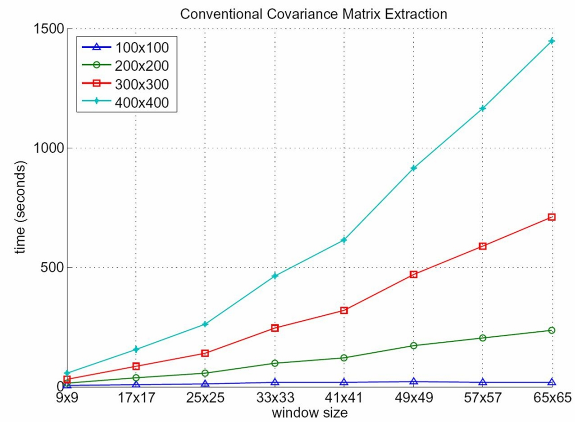
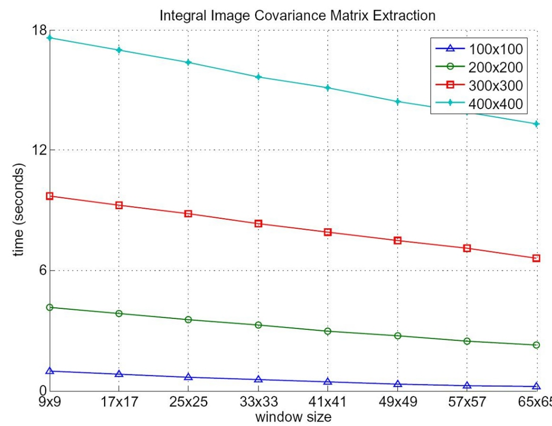


(a) Conventional approach



(b) Proposed method

Fig. 4. Computation times of covariance extraction for fixed number of features.

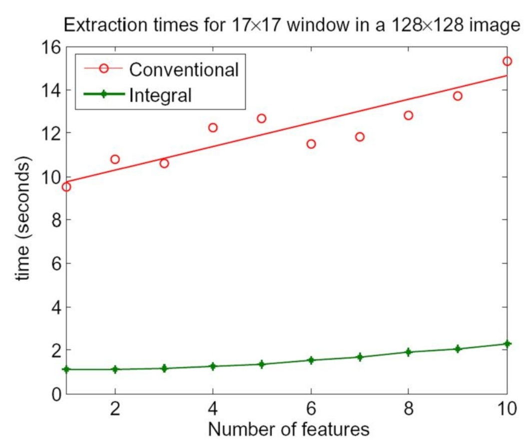


Fig. 5. Proposed method becomes more advantageous with increasing number of features.

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