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Abstract

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Effect of Timing Jitter on the Trade-off Between Processing Gains[1]

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***Abstract*— In time hopping impulse radio, N_f pulses of duration T_c are transmitted for each symbol. This gives rise to two types of processing gain: (i) pulse combining gain, which is a factor N_f, and (ii) pulse spreading gain, which is $N_c = T_f/T_c$, where T_f is the mean interval between two subsequent pulses. This paper investigates the tradeoff between these two types of processing gain with and without random polarity codes in the presence of timing jitters. Bit error rate expressions are derived for both coded and uncoded systems and are used as the criterion to choose optimal N_f and N_c values. The effect of timing jitters and multiple access interference on the selection of optimal system parameters are explained through theoretical analysis. Simulation studies support the theoretical results.**

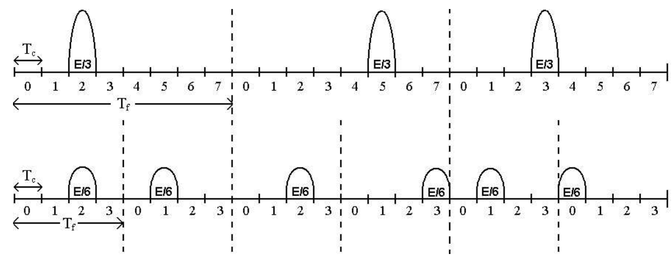

Fig. 1. Two different cases for an uncoded BPSK-modulated TH-IR system when $N = 24$. For the first case, $N_c = 8$, $N_f = 3$, pulse energy is $E/3$ and for the second case $N_c = 4$, $N_f = 6$, pulse energy is $E/6$.

I. Introduction

Recently, communication systems that employ ultra-wideband (UWB) signals have drawn considerable attention. UWB signals occupy a bandwidth larger than 500MHz and can make use of the existing spectrum. Recent Federal Communications Commission (FCC) rulings [4, 5] specify the regulations for UWB communication systems in the US. Similar rulings are expected in the near future for Europe and Japan.

Commonly, impulse radio (IR) systems, which transmit very short pulses with a low duty cycle, are employed to implement UWB systems [6]. In an IR system, a train of pulses is sent and information is usually conveyed by the position or the polarity of the pulses, which correspond to Pulse Position Modulation (PPM) and Binary Phase Shift Keying (BPSK), respectively. Also, in order to prevent catastrophic collisions among different users and thus provide robustness against multiple access interference, each information symbol is represented not by one pulse but by a sequence of pulses and the location of the pulses within the sequence is determined by a pseudo-random time-hopping (TH) sequence [6]. For example, the first signal in Figure 1 is an uncoded[3] BPSK-modulated TH-IR signal where 3 pulses are sent in order to represent one bit ($+1$ in this case) and the pulse positions are determined by the TH sequence $\{2, 5, 3\}$.

The number of pulses that are sent for each information symbol is denoted by N_f. This first type of processing gain is called the pulse combining gain, which is the pulse rate of the system. The second type of processing gain N_c is the pulse spreading gain and is defined as the ratio of average time between the two consecutive transmissions and the actual transmission time, that is, $N_c = T_f/T_c$. The total processing gain is defined as $N = N_c N_f$ and assumed to be constant and large [2]. The aim of this paper is to investigate the trade-off between the two types of processing gain, N_c and N_f, and to calculate the optimal N_c (N_f) value such that bit error rate of the system is minimized[4]. In other words, the problem is to decide whether or not sending more pulses each with less energy is more desirable in terms of bit error rate performances than sending fewer pulses each with more energy (Figure 1).

This problem is originally investigated in [2]. Also [3] analyzed the problem from an information theoretic point of view for the single-user case. In [2], it is concluded that in multiuser flat fading channels, the system performance is independent of the pulse rate for a coded system and it is in favor of small pulse rates for an uncoded system. However, no timing jitters are considered in that work. As we will see in this paper, timing jitters have an effect on the trade-off between the processing gains and they modify the dependency of the bit error rate expressions to processing gain parameters. In this paper, the trade-off between two types of processing gain is investigated in the presence of timing jitters and expressions for the bit error rate are derived for both coded and uncoded systems.

The remainder of the paper is organized as follows. Section II describes the transmitted signal model and components of the received signal at the output of a Matched Filter (MF) receiver. The BER expressions for coded and uncoded systems are derived in Sections III and IV, respectively and the trade-

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[3]In coded systems, the polarity of all pulses are determined by a random polarity code sequence.

[4]FCC also imposes restriction on peak-to-average ratio (PAR), which is not considered in this paper.

off between the processing gains is investigated. Section V presents some simulation studies and numerical examples and finally Section VI concludes the paper.

II. Signal Model

Consider a binary phase-shift keyed random time-hopping impulse-radio (TH-IR) system where the transmitted signal from user k in an N_u-user setting is represented by the following model:

$$s_{tx}^{(k)}(t)=\sqrt{\frac{E_k}{N_f}}\sum_{j=-\infty}^{\infty}d_j^{(k)}\,b_{\lfloor j/N_f\rfloor}^{(k)}w_{tx}(t-jT_f-c_j^{(k)}T_c+\epsilon_j^{(k)}),\tag{1}$$

where w_{tx} is the transmitted unit-energy pulse, E_k is the bit energy of user k, $\epsilon_j^{(k)}$ is the timing jitter at jth pulse of the kth user, T_f is the average pulse repetition time (also called the "frame" time), N_f is the number of pulses representing one information symbol, which is called the pulse rate of the system, and $b_{\lfloor j/N_f\rfloor}^{(k)}\in\{+1,-1\}$ is the information symbol transmitted by user k. In order to allow the channel to be exploited by many users and avoid catastrophic collisions, a pseudo-random sequence $\{c_j^{(k)}\}$, where $c_j^{(k)}\in\{0,1,...,N_c-1\}$, is assigned to each user. This sequence is called time hopping sequence and provides an additional time shift of $c_j^{(k)}T_c$ seconds to the jth pulse of the kth user where T_c is the chip interval and is chosen to satisfy $T_c\leq T_f/N_c$ in order to prevent the pulses from overlapping. Without loss of generality, $T_f=N_cT_c$ is assumed throughout the paper.

Two different IR systems will be considered depending on $d_j^{(k)}$. The system is called "uncoded" if $d_j^{(k)}=1,\ \forall k,j$, and it is called "coded" if $d_j^{(k)}$ are binary random variables taking values ±1 with equal probability and are independent for $(k,j)\neq(l,i)$.

$N=N_cN_f$ is defined to be the total processing gain. Assuming a large and constant N value, the aim is to obtain the optimal N_c (N_f) value that minimizes the bit error rate (BER) of the system.

The received signal over a flat fading channel (for multipath channels, see [1]) in an N_u-user system can be expressed as

$$\begin{aligned}r(t)&=\sum_{k=1}^{N_u}\sqrt{\frac{E_k}{N_f}}\sum_{j=-\infty}^{\infty}d_j^{(k)}\,b_{\lfloor j/N_f\rfloor}^{(k)}\\&\times\ w_{rx}(t-jT_f-c_j^{(k)}T_c+\epsilon_j^{(k)})+\sigma_n n(t),\end{aligned}\tag{2}$$

where w_{rx} is the received UWB pulse and $n(t)$ is a zero mean white Gaussian noise with unit spectral density.

We assume that all users are synchronized[5] and jitters are independent identically distributed (i.i.d.) for each user. That is, $\epsilon_j^{(k)}$ for $j=\ldots,-1,0,1,\ldots$ form an i.i.d. sequence. Also the jitters are assumed to be small compared to the chip interval T_c.

Considering a Matched Filter (MF) receiver, the template signal at the receiver can be expressed as follows:

$$s_{temp}^{(1)}(t)=\frac{1}{\sqrt{N_f}}\sum_{j=iN_f}^{(i+1)N_f-1}d_j^{(1)}w_{rx}(t-jT_f-c_j^{(1)}T_c),\tag{3}$$

[5]Symbol synchronization among different users does not need not be assumed for coded systems. Chip synchronization is sufficient in that case.

where, without loss of generality, user 1 is assumed to be the user of interest. Also note that no timing jitters are considered for the template signal since the jitter models in the received signal are assumed to account for those jitters as well.

From (2) and (3), the MF output for user 1 can be expressed as follows:

$$y_1=\frac{\sqrt{E_1}}{N_f}\,b_i^{(1)}\sum_{j=iN_f}^{(i+1)N_f-1}R(\epsilon_j^{(1)})+a+n,\tag{4}$$

where the first term is the signal part of the output with $R(x)=\int_{-\infty}^{\infty}w_{rx}(t)w_{rx}(t-x)dt$ being the symmetric autocorrelation function of the UWB pulse, a is the multiple access interference (MAI) due to other users and n is the output noise, $n\sim\mathcal{N}(0,\,\sigma_n^2)$.

The MAI term can be expressed as sum of interference terms from each user, that is, $a=\sum_{k=2}^{N_u}a^{(k)}$, where each interference term is in turn the summation of interference due to one pulse of the template signal:

$$a^{(k)}=\frac{\sqrt{E_k}}{N_f}\sum_{l=iN_f}^{(i+1)N_f-1}a_l^{(k)},\tag{5}$$

where

$$\begin{aligned}a_l^{(k)}&=d_l^{(1)}\int w_{rx}(t-lT_f-c_l^{(1)}T_c)\sum_{j=-\infty}^{\infty}d_j^{(k)}b_{\lfloor j/N_f\rfloor}^{(k)}\\&\times\ w_{rx}(t-jT_f-c_j^{(k)}T_c+\epsilon_j^{(k)})\,dt.\end{aligned}\tag{6}$$

As can be seen from (6), $a_l^{(k)}$ denotes the interference from user k to the lth pulse of the template signal.

Let $p_l^{(1)}$ denote the position of the lth pulse of the template signal in the lth frame ($p_l^{(1)}=1,...,N_c$) for $l=iN_f,...,(i+1)N_f-1$. Similarly, write $p_l^{(k)}$ for the position of the lth pulse of the received signal from user k. Then, $a_l^{(k)}$ can be expressed as follows for $p_l^{(1)}=2,...,N_c-1$:

$$\begin{aligned}a_l^{(k)}=b_i^{(k)}d_l^{(1)}d_l^{(k)}\,[\,&R(\epsilon_l^{(k)})\,I_{\{p_l^{(1)}=p_l^{(k)}\}}\\&+R(T_c-\epsilon_l^{(k)})\,I_{\{p_l^{(1)}-p_l^{(k)}=1\}}I_{\{\epsilon_l^{(k)}>0\}}\\&+R(T_c+\epsilon_l^{(k)})\,I_{\{p_l^{(k)}-p_l^{(1)}=1\}}I_{\{\epsilon_l^{(k)}<0\}}],\end{aligned}\tag{7}$$

for $l=iN_f,...,(i+1)N_f-1$, where I_A is the indicator function taking value 1 in set A and 0 outside. In obtaining (7), the following observation is employed: There occurs interference from user k to the lth pulse of the template signal if user k has its lth pulse at the same position as the lth pulse of the template signal or it has its lth pulse at a neighboring position to lth pulse of the template signal and there is a partial overlap due to the effect of timing jitter.

For $p_l^{(1)}=1$, we also consider the interference from the previous frame of the signal received from user k:

$$\begin{aligned}a_l^{(k)}=b_i^{(k)}d_l^{(1)}d_l^{(k)}[&R(\epsilon_l^{(k)})I_{\{p_l^{(k)}=1\}}\\&+R(T_c+\epsilon_l^{(k)})I_{\{p_l^{(k)}=2\}}I_{\{\epsilon_l^{(k)}<0\}}]\\+b_i^{(k)}d_l^{(1)}d_{l-1}^{(k)}&R(T_c-\epsilon_{l-1}^{(k)})I_{\{p_{l-1}^{(k)}=N_c\}}I_{\{\epsilon_{l-1}^{(k)}>0\}},\end{aligned}\tag{8}$$

for $l = iN_f + 1, ..., (i+1)N_f - 1$. Note that for $l = iN_f$, we just need to replace $b_i^{(k)}$ in the third term by $b_{i-1}^{(k)}$ since the previous bit will be in effect in that case.

Similarly, for $p_l^{(1)} = N_c$,

$$\begin{aligned} a_l^{(k)} &= b_i^{(k)} d_l^{(1)} d_l^{(k)} [R(\epsilon_l^{(k)}) I_{\{p_l^{(k)}=N_c\}} \\ &\quad + R(T_c - \epsilon_l^{(k)}) I_{\{p_l^{(k)}=N_c-1\}} I_{\{\epsilon_l^{(k)}>0\}}] \\ &\quad + b_i^{(k)} d_l^{(1)} d_{l+1}^{(k)} R(T_c + \epsilon_{l+1}^{(k)}) I_{\{p_{l+1}^{(k)}=1\}} I_{\{\epsilon_{l+1}^{(k)}<0\}}, \end{aligned} \tag{9}$$

for $l = iN_f, ..., (i+1)N_f - 2$. For $l = (i+1)N_f - 1$, $b_i^{(k)}$ in the third term is replaced by $b_{i+1}^{(k)}$.

Our aim is to obtain the probability distribution of $a^{(k)} = \frac{\sqrt{E_k}}{N_f} \sum_{l=iN_f}^{(i+1)N_f-1} a_l^{(k)}$. We will consider coded and uncoded systems separately at this point.

III. Coded Systems

For coded systems, the following lemma approximates the probability distribution of the MAI from user k, $a^{(k)}$:

Lemma 3.1: *Assume that $N \longrightarrow \infty$ and $\frac{N_f}{N_c} \longrightarrow c > 0$. Then, the MAI from user k, $a^{(k)}$, is asymptotically normally distributed as*

$$a^{(k)} \sim \mathcal{N}(0\,,\, E_k \gamma_2 / N), \tag{10}$$

where $\gamma_2 = \mathrm{E}\{R^2(\epsilon^{(k)})\} + \mathrm{E}\{R^2(T_c - |\epsilon^{(k)}|)\}$[6].

Proof *See Appendix A.*

When probability distributions of the jitters are i.i.d. for all users, the MAI term a is distributed as follows:

$$a \sim \mathcal{N}\left(0\,,\, \frac{\gamma_2}{N} \sum_{k=2}^{N_u} E_k\right). \tag{11}$$

Assuming interferers with equal energy, for simplicity, we have $a \sim \mathcal{N}(0\,,\,(N_u - 1)E\gamma_2/N)$. Then, using (4), the bit error rate (BER) conditioned on the timing jitters of user 1 can be expressed as follows:

$$P_{e|\underline{\epsilon}^{(1)}} = Q\left(\frac{\frac{\sqrt{E_1}}{N_f} \sum_{j=iN_f}^{(i+1)N_f-1} R(\epsilon_j^{(1)})}{\sqrt{\sigma_n^2 + (N_u - 1)E\gamma_2/N}}\right), \tag{12}$$

where $\underline{\epsilon}^{(1)} = [\epsilon_{iN_f}^{(1)} \ldots \epsilon_{(i+1)N_f-1}^{(1)}]$.

For large value of N_f, it follows from the Central Limit Theorem (CLT) that $\frac{1}{N_f} \sum_{j=iN_f}^{(i+1)N_f-1} R(\epsilon_j^{(1)})$ is approximately Gaussian. That is,

$$\frac{1}{N_f} \sum_{j=iN_f}^{(i+1)N_f-1} R(\epsilon_j^{(1)}) \sim \mathcal{N}(\mu, \sigma^2/N_f), \tag{13}$$

where $\mu = \mathrm{E}\{R(\epsilon_j^{(1)})\}$ and $\sigma^2 = \mathrm{Var}\{R(\epsilon_j^{(1)})\}$. Then, using the relation $\mathrm{E}\{Q(X)\} = Q\left(\frac{\hat{\mu}}{\sqrt{1+\hat{\sigma}^2}}\right)$ for $X \sim \mathcal{N}(\hat{\mu}, \hat{\sigma}^2)$, the unconditional BER can be expressed as follows:

$$P_e = Q\left(\frac{\sqrt{E_1}\mu}{\sqrt{\frac{E_1\sigma^2}{N} N_c + (N_u - 1)E\gamma_2/N + \sigma_n^2}}\right). \tag{14}$$

[6]Note that the frame index is removed from the jitter for simplicity since jitters of a user at different frames are i.i.d.

From the last expression, it is observed that BER increases as N_c increases. In other words, BER is smaller for larger pulse rate N_f. It is seen from Lemma 3.1 that MAI for a coded system is asymptotically independent of processing gains. Therefore, the second term in the denominator of (14), which is the term due to MAI, does not depend on N_c (N_f). The only term that depends on the processing gain is the first term in the denominator, which reflects the effect of timing jitters. This effect is mitigated by sending more pulses per bit (large N_f) as can be observed from (13). Therefore, for a coded system, keeping N_f large helps to reduce BER. Also note that in the absence of timing jitters, (14) reduces to $P_e = Q\left(\frac{\sqrt{E_1}}{\sqrt{(N_u-1)E/N+\sigma_n^2}}\right)$, in which case there is no effect of processing gain parameters to BER performance, as stated in [2].

IV. Uncoded Systems

For uncoded systems, the following lemma approximates the probability distribution of the MAI from user k, $a^{(k)}$:

Lemma 4.1: *Assume that $N \longrightarrow \infty$ and $\frac{N_f}{N_c} \longrightarrow c > 0$. Then, the MAI from user k, $a^{(k)}$, given the information bit $b_i^{(k)}$, is approximately distributed as*

$$a^{(k)} | b_i^{(k)} \sim \mathcal{N}\left(b_i^{(k)} \frac{\gamma_1 \sqrt{E_k}}{N_c}\,,\, \frac{E_k}{N}(\gamma_2 - \frac{\gamma_1^2}{N_c} + \frac{\beta_1}{N_c^2} + \frac{\beta_2}{N_c^3})\right), \tag{15}$$

where

$$\begin{aligned} \gamma_1 &= \mathrm{E}\{R(\epsilon^{(k)})\} + \mathrm{E}\{R(T_c - |\epsilon^{(k)}|)\}, \\ \gamma_2 &= \mathrm{E}\{R^2(\epsilon^{(k)})\} + \mathrm{E}\{R^2(T_c - |\epsilon^{(k)}|)\}, \\ \beta_1 &= 2\mathrm{E}\{R(T_c - |\epsilon^{(k)}|)R(\epsilon^{(k)})\} - 2(\mathrm{E}\{R(T_c - |\epsilon^{(k)}|)\})^2 \\ &+ 4\int_{-\infty}^{0} R(T_c + \epsilon^{(k)}) p(\epsilon^{(k)}) d\epsilon^{(k)} \\ &\times \int_{0}^{\infty} R(T_c - \epsilon^{(k)}) p(\epsilon^{(k)}) d\epsilon^{(k)}, \\ \beta_2 &= 2(\mathrm{E}\{R(T_c - |\epsilon^{(k)}|)\})^2. \end{aligned} \tag{16}$$

Proof *See Appendix B.*

Note that for systems with large N_c, the MAI in the uncoded case given the information symbol $b_i^{(k)}$ can be approximately expressed as $a^{(k)}|b_i^{(k)} \sim \mathcal{N}\left(b_i^{(k)}\gamma_1\sqrt{E_k}/N_c\,,\, \frac{E_k}{N}(\gamma_2 - \gamma_1^2/N_c)\right)$.

First consider a two-user system. For equiprobable information symbols ± 1, the BER conditioned on timing jitters of the first user can be shown to be

$$\begin{aligned} \mathrm{P}_{e|\underline{\epsilon}^{(1)}} &\approx \frac{1}{2} Q\left(\frac{\frac{\sqrt{E_1}}{N_f} \sum_{j=iN_f}^{(i+1)N_f-1} R(\epsilon_j^{(1)}) + \frac{\sqrt{E_2}}{N_c}\gamma_1}{\sqrt{\sigma_n^2 + \frac{E_2}{N}(\gamma_2 - \gamma_1^2/N_c)}}\right) \\ &+ \frac{1}{2} Q\left(\frac{\frac{\sqrt{E_1}}{N_f} \sum_{j=iN_f}^{(i+1)N_f-1} R(\epsilon_j^{(1)}) - \frac{\sqrt{E_2}}{N_c}\gamma_1}{\sqrt{\sigma_n^2 + \frac{E_2}{N}(\gamma_2 - \gamma_1^2/N_c)}}\right). \end{aligned} \tag{17}$$

Then, for large N_f values, we can again invoke the CLT and obtain $\frac{1}{N_f} \sum_{j=iN_f}^{(i+1)N_f-1} R(\epsilon_j^{(1)}) \sim \mathcal{N}(\mu, \sigma^2/N_f)$ with $\mu = \mathrm{E}\{R(\epsilon_j^{(1)})\}$ and $\sigma^2 = \mathrm{Var}\{R(\epsilon_j^{(1)})\}$. Then, the unconditional

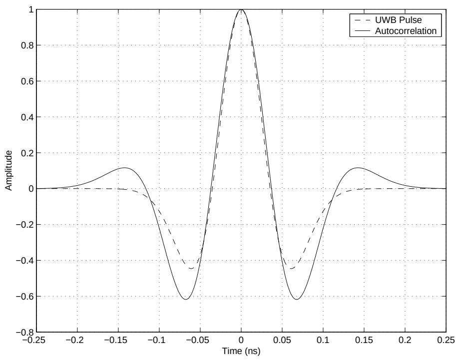

Fig. 2. UWB pulse and the autocorrelation function for $T_c = 0.25ns$.

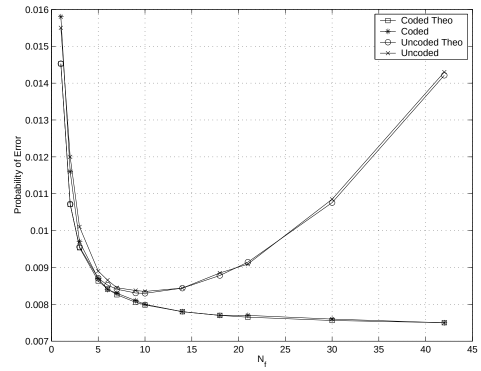

Fig. 3. BER versus N_f for coded and uncoded systems.

BER can be similarly obtained as

$$P_e \approx \frac{1}{2} Q \left(\frac{\sqrt{E_1}\mu + \frac{\sqrt{E_2}}{N_c}\gamma_1}{\sqrt{\frac{E_1\sigma^2}{N}N_c + \frac{E_2}{N}(\gamma_2 - \gamma_1^2/N_c) + \sigma_n^2}} \right) + \frac{1}{2} Q \left(\frac{\sqrt{E_1}\mu - \frac{\sqrt{E_2}}{N_c}\gamma_1}{\sqrt{\frac{E_1\sigma^2}{N}N_c + \frac{E_2}{N}(\gamma_2 - \gamma_1^2/N_c) + \sigma_n^2}} \right). \quad (18)$$

For the multiuser case, assume that all interfering users have the same energy E and probability distributions of the jitters are i.i.d. for all of them. Then, the total MAI can be approximated by a zero mean Gaussian random variable with variance $(N_u - 1)\left[\frac{E}{N_c^2}\gamma_1^2 + \frac{E}{N}(\gamma_2 - \gamma_1^2/N_c)\right]$ for sufficiently large number of users, N_u.

Then, after similar manipulations, BER can be expressed as

$$P_e \approx Q \left(\frac{\sqrt{E_1}\mu}{\sqrt{\frac{E_1\sigma^2}{N}N_c + (N_u - 1)E\left(\frac{\gamma_2}{N} + \frac{\gamma_1^2}{N_c^2} - \frac{\gamma_1^2}{NN_c}\right) + \sigma_n^2}} \right). \quad (19)$$

Considering (19), it is seen that for relatively small N_c values, the second term in the denominator, which is the term due to MAI, becomes large and causes an increase in the BER. Similarly, when N_c is large, the first term in the denominator becomes significant and the BER becomes high again. Therefore, we expect to have an optimal N_c value. Intuitively, for small N_c values, the number of pulses per bit, N_f, is large. Therefore, we have high BER due to large amount of MAI. As N_c becomes large, MAI becomes more negligible. However, making N_c very large again causes an increase in BER since N_f becomes small in that case and effect of timing jitter becomes more significant. The optimal N_c (N_f) value can be approximated by equating the first derivative of (19) with respect to N_c to zero.

V. Simulation Results

In this section, bit error rate (BER) performances of coded and uncoded systems are simulated for different values of processing gains and the results are compared to the theoretical results. The UWB pulse[7] and the normalized autocorrelation function used in the simulations are as follows [7]:

$$w(t) = \left(1 - \frac{4\pi t^2}{\tau^2}\right) e^{-2\pi t^2/\tau^2}, \quad (20)$$

$$R(\Delta t) = \left[1 - 4\pi(\frac{\Delta t}{\tau})^2 + \frac{4\pi^2}{3}(\frac{\Delta t}{\tau})^4\right] e^{-\pi(\frac{\Delta t}{\tau})^2}, \quad (21)$$

where $\tau = 0.125ns$ is used.

The timing jitter is modelled by $\mathcal{U}[-25ps, 25ps]$ and T_c is chosen to be $0.25ns$. The total processing gain $N = N_c N_f$ is taken to be 630. Also all 10 users ($N_u = 10$) are assumed to be sending unit energy per bit ($E_k = 1\ \forall k$) and $\sigma_n^2 = 0.1$.

Figure 3 shows the BER of the coded and the uncoded system for different N_f values. It is seen that theoretical values match quite closely with the simulation results, especially when N_f gets larger, since the Gaussian approximation gets better as N_f increases. For the coded system, the BER decreases as N_f increases. Since the MAI is asymptotically independent of N_f as shown in Lemma 3.1, the only effect to consider will be timing jitters. Since the effect of timing jitter is reduced for large N_f, the plots for coded system show a decrease in BER as N_f increases. For the uncoded system, there is an optimal value of the processing gain that minimizes the BER of the system. In this case, there are both the effect of timing jitters and the effect of MAI. The effect of timing jitters is mitigated using large N_f while that of MAI is reduced using small N_f. The optimal value of the processing gains can be approximately calculated using (19).

VI. Conclusion

The trade-off between two types of processing gain is investigated in the presence of timing jitters. It is concluded that in a flat fading channel sending more pulses per bit decreases the bit error rate in a coded system since MAI is independent of processing gains and effect of timing jitters is reduced by sending more pulses. In an uncoded system, there is a trade-off between N_c and N_f, which reflects the effects of timing jitter and MAI. Optimal processing gains can be found by using an approximate closed form expression for the bit error rate. Future work will extend the results to frequency selective channels.

[7] $w_{rx}(t) = w(t)/\sqrt{E_p}$ with $E_p = \int_{-\infty}^{\infty} w^2(t)dt$ is used as the received UWB pulse with unit energy.

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APPENDIX

A. Proof of Lemma 3.1

Consider $a_l^{(k)}$ given by equations (7)-(9). In the coded case, it is easy to see that $\mathrm{E}\{a_l^{(k)}\} = 0$ due to the independence of polarity codes for different frame indices. To calculate the variance, the relation $\mathrm{E}\{(a_l^{(k)})^2\} = \mathrm{E}\{\mathrm{E}\{(a_l^{(k)})^2|\underline{\epsilon}^{(k)}\}\}$ is employed. From equations (7)-(9), we have

$$\begin{aligned}\mathrm{E}\{(a_l^{(k)})^2|\underline{\epsilon}^{(k)}\} = \frac{N_c-2}{N_c^2}[R^2(\epsilon_l^{(k)}) + R^2(T_c-\epsilon_l^{(k)})\, I_{\{\epsilon_l^{(k)}>0\}} \\ +R^2(T_c+\epsilon_l^{(k)})\, I_{\{\epsilon_l^{(k)}<0\}}] + \frac{1}{N_c^2}[R^2(\epsilon_l^{(k)}) \\ +R^2(T_c+\epsilon_l^{(k)})I_{\{\epsilon_l^{(k)}<0\}} + R^2(T_c-\epsilon_{l-1}^{(k)})I_{\{\epsilon_{l-1}^{(k)}>0\}}] \\ +\frac{1}{N_c^2}[R^2(\epsilon_l^{(k)}) + R^2(T_c-\epsilon_l^{(k)})I_{\{\epsilon_l^{(k)}>0\}} \\ +R^2(T_c+\epsilon_{l+1}^{(k)})I_{\{\epsilon_{l+1}^{(k)}<0\}}],\end{aligned}$$

where independence of the polarity codes for different frame indices and the fact that probability that a pulse is in a given chip in the frame is $1/N_c$ are employed.

Then, it is straightforward to show that

$$\mathrm{E}\{(a_l^{(k)})^2\} = \gamma_2/N_c, \tag{22}$$

where $\gamma_2 = \mathrm{E}\{R^2(\epsilon_l^{(k)})\} + \mathrm{E}\{R^2(T_c - |\epsilon_l^{(k)}|)\}$.

Note that $a_{iN_f}^{(k)}, ..., a_{(i+1)N_f-1}^{(k)}$ are identically distributed but not independent. However, they form a 1-dependent sequence [8]. Therefore, for large N_f values, sum of them converge to a zero mean Gaussian random variable with variance $N_f[\mathrm{E}\{(a_{iN_f}^{(k)})^2\} + 2\mathrm{E}\{a_{iN_f}^{(k)}a_{iN_f+1}^{(k)}\}]$ [8]. It is easy to show that the cross-correlation term is zero using the independence of polarity codes for different indices. Hence,

$$\sum_{l=iN_f}^{(i+1)N_f-1} a_l^{(k)} \sim \mathcal{N}(0\,,\, \gamma_2 N_f/N_c). \tag{23}$$

Then, using (5), we get $a^{(k)} \sim \mathcal{N}(0\,,\, E_k\gamma_2/N)$.

B. Proof of Lemma 4.1

Similar to the proof of the previous lemma, we want to approximate the distribution of $a^{(k)} = \frac{\sqrt{E_k}}{N_f}\sum_{l=iN_f}^{(i+1)N_f-1} a_l^{(k)}$ given the information bit $b_i^{(k)}$ by a Gaussian random variable. However, in this case, $a_{iN_f}^{(k)}, ..., a_{(i+1)N_f-1}^{(k)}$ are not identically distributed due to the possible small difference for the edge values $a_{iN_f}^{(k)}$ and $a_{(i+1)N_f-1}^{(k)}$ as stated after equations (8) and (9). However, those differences can be neglected assuming large N_c and N_f values. Then, $a_{iN_f}^{(k)}, ..., a_{(i+1)N_f-1}^{(k)}$ can be considered as identically distributed. The mean value can be calculated using $\mathrm{E}\{a_l^{(k)}|b_i^{(k)}\} = \mathrm{E}\{\mathrm{E}\{a_l^{(k)}|\underline{\epsilon}^{(k)}, b_i^{(k)}\}\}$. From equations (7)-(9), we get

$$\begin{aligned}\mathrm{E}\{a_l^{(k)}|\underline{\epsilon}^{(k)}, b_i^{(k)}\} = [R(\epsilon_l^{(k)}) + R(T_c-\epsilon_l^{(k)})\, I_{\{\epsilon_l^{(k)}>0\}} \\ +R(T_c+\epsilon_l^{(k)})\, I_{\{\epsilon_l^{(k)}<0\}}]\frac{N_c-2}{N_c^2}b_i^{(k)} + [R(\epsilon_l^{(k)}) \\ +R(T_c+\epsilon_l^{(k)})I_{\{\epsilon_l^{(k)}<0\}} + R(T_c-\epsilon_{l-1}^{(k)})I_{\{\epsilon_{l-1}^{(k)}>0\}}]\frac{b_i^{(k)}}{N_c^2} \\ +[R(\epsilon_l^{(k)}) + R(T_c-\epsilon_l^{(k)})I_{\{\epsilon_l^{(k)}>0\}} \\ +R(T_c+\epsilon_{l+1}^{(k)})I_{\{\epsilon_{l+1}^{(k)}<0\}}]\frac{b_i^{(k)}}{N_c^2}.\end{aligned} \tag{24}$$

Then, taking expectation with respect to timing jitters, we get

$$\mathrm{E}\{a_l^{(k)}|b_i^{(k)}\} = b_i^{(k)}\gamma_1/N_c, \tag{25}$$

where $\gamma_1 = \mathrm{E}\{R(\epsilon_l^{(k)})\} + \mathrm{E}\{R(T_c - |\epsilon_l^{(k)}|)\}$.

By similar calculations, it can be shown that

$$\begin{aligned}\mathrm{E}\{(a_l^{(k)})^2|b_i^{(k)}\} = \frac{\gamma_2}{N_c} + \frac{2}{N_c^3}\mathrm{E}\{R(\epsilon^{(k)})\}\mathrm{E}\{R(T_c-|\epsilon^{(k)}|)\} \\ +\frac{4}{N_c^3}\int_{-\infty}^{0} R(T_c+\epsilon^{(k)})p(\epsilon^{(k)})d\epsilon^{(k)}\int_0^\infty R(T_c-\epsilon^{(k)})p(\epsilon^{(k)})d\epsilon^{(k)},\end{aligned} \tag{26}$$

where $p(\epsilon^{(k)})$ is the probability density function of i.i.d. timing jitters for user k and $\gamma_2 = \mathrm{E}\{R^2(\epsilon_l^{(k)})\} + \mathrm{E}\{R^2(T_c - |\epsilon_l^{(k)}|)\}$. Note that frame indices are omitted in the last equation since the results do not depend on the frame index.

The cross-correlations between consecutive values of the 1-dependent sequence $a_{iN_f}^{(k)}, ..., a_{(i+1)N_f-1}^{(k)}$ can be obtained as

$$\begin{aligned}\mathrm{E}\{a_l^{(k)}a_{l+1}^{(k)}|b_i^{(k)}\} = \gamma_1^2/N_c^2 - \frac{1}{N_c^3}\gamma_1\mathrm{E}\{R(T_c-|\epsilon^{(k)}|)\} \\ +\frac{1}{N_c^3}\mathrm{E}\{R(T_c-|\epsilon^{(k)}|)R(\epsilon^{(k)})\} + \frac{1}{N_c^4}(\mathrm{E}\{R(T_c-|\epsilon^{(k)}|)\})^2.\end{aligned}$$

Then, invoking the theorem for 1-dependent sequences [8], the sum of interferences due to each pulse of the template, $\sum_{l=iN_f}^{(i+1)N_f-1} a_l^{(k)}|b_i^{(k)}$, is distributed as a Gaussian random variable with mean $N_f\mathrm{E}\{a_l^{(k)}|b_i^{(k)}\}$ and variance $N_f\left(\mathrm{Var}\{a_l^{(k)}|b_i^{(k)}\} + 2[\mathrm{E}\{a_l^{(k)}a_{l+1}^{(k)}|b_i^{(k)}\} - (\mathrm{E}\{a_l^{(k)}|b_i^{(k)}\})^2]\right)$, which can be expressed as

$$\mathcal{N}\left(\frac{b_i^{(k)}N_f\gamma_1}{N_c}\,,\, N_f(\frac{\gamma_2}{N_c} - \frac{\gamma_1^2}{N_c^2} + \frac{\beta_1}{N_c^3} + \frac{\beta_2}{N_c^4})\right), \tag{27}$$

where γ_1, γ_2, β_1 and β_2 are as in (16).

Then, using (27) in (5), (15) is obtained.